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OPTIMAL LINEAR CONTROL

(CHARACTERIZATION AND LOOP TRANSMISSION PROPERTIES OF MULTIVARIABLE SYSTEMS)

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This report presents results on characterizations of multivariable control systems. The major results are: (1) extension of the asymptotic modal approach of selecting quadratic performance indexes to the discrete case; (2) the lack of guaranteed stability margins for linear quadratic Gaussian systems, and a design procedure to improve the stability margins for such systems; (3) characterization of multivariable loop transmission properties in terms of singular values and singular vectors, which provides a meaningful concept for multivariable stability margins, and (4) a method for selecting compensation → next page

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20. structures to achieve desired loop transmission properties.

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SECTION 1

INTRODUCTION AND SUMMARY

The goal of this research program is the advancement of linear control design technology. The initial focus of the program was to advance linear optimal design techniques so that they would yield operational control laws that met conventional design specifications. The development of design techniques with the assumption that full state feedback is available was reported for single input system [1] and for multi-input systems in [2]. The significant result for multi-input systems reported in [2] was the characterization of quadratic weighting matrices in terms of their asymptotic modal properties (eigenvalues and eigenvectors of the optimally controlled system), and a procedure for constructing quadratic weighting matrices based on desired modal characteristics. Direct extensions of these results to discrete systems are described in this report along with development of asymptotic expansions for the control law and the eigenvalues and eigenvectors of the optimal system.

To meet conventional specifications, it is often necessary to add dynamic compensators to the optimal designs. Issues concerning compensation to achieve desired high frequency attenuation and system bandwidth were investigated, and it was found that compensators reduce stability margins. Noting that Kalman filters have features in common with compensators and recognizing that operational control systems would often lack the luxury of full state measurement so that some form of state estimation must be

incorporated, the stability margins of linear quadratic Gaussian (LQG) system were examined. This led to the following major results:

- LQG controllers have no guaranteed stability margins. This is in contrast to the 6 dB and 60 degree phase margins guaranteed for linear optimal controllers using state feedback.

- Introduction of an additional Kalman filter design parameter in the synthesis of LQG controllers leads to improved stability margins which can be made arbitrarily close to those of the full state feedback controllers.

These results led to further investigation of the loop transmission properties of multivariable systems. A meaningful characterization of these properties in terms of the singular values and singular vectors of the system was developed which provides a natural generalization of the scalar stability margin concept for single input/single output systems. This characterization was used to develop compensation structures for achieving desired transmission loop properties. The characterization also provides a potentially useful alternate method for selecting quadratic weights for the linear quadratic state feedback (LQ) design problem.

The refinements of the characterization of linear optimal controllers on the basis of asymptotic modal properties is described in Section 2 and Appendixes A and B. The results of the investigation of the loop transmission properties are presented in Section 3 and Appendixes C through I.

SECTION 2

REFINEMENTS OF THE ASYMPTOTIC MODAL PROPERTIES OF LINEAR OPTIMAL CONTROLLERS

In reference [2] quadratic performance indices were characterized in terms of asymptotic eigenvalues and eigenvectors. That is, for an n^{th} order system with m inputs

$$\dot{x} = Fx + Gu \tag{1}$$

the quadratic performance index

$$J = \int_0^\infty (x^T Q x + \rho^2 u^T R u)\, dt \tag{2}$$

may be characterized by the asymptotic behavior of the closed-loop eigenvalues and eigenvectors as the parameter ρ tends to zero. Furthermore, a procedure was described in [2] for defining the weighting matrices, Q and R, on the basis of desired asymptotic eigenvalues and eigenvectors in the case where n-m of the desired asymptotic eigenvalues are finite.

An alternative approach to defining the weighting matrices on the basis of desired asymptotic modal properties was developed which is applicable to the above case and to the case in which there are more than n-m asymptotic eigenvalues that are finite. This alternative approach also yields the two leading terms in the asymptotic expansion of the optimal control law. The details of this approach are given in Appendix A.

A method was also developed for finding the leading terms in the asymptotic expansions of the eigenvalues and eigenvectors of the optimal system as well as the gain matrix of the optimal control system. This method as described in Appendix B may be used to analyze the asymptotic properties of any given quadratic weighting matrices.

Reference [3] provides a further generalization of the asymptotic modal properties to the case where there are less than n-m asymptotic finite eigenvalues with the remaining eigenvalues tending to infinity in selectable Butterworth patterns.

In addition to the above developments, the characterization of performance indices given in [2] for continuous systems was extended to discrete systems as follows. Consider the controllable discrete system

$$x_{k+1} = Ax_k + Bu_k, \; x_o \text{ given} \tag{3}$$

where x_k is an n-vector, u_k is an m-vector with $m \leq n$, and the rank of B is m. The optimal quadratic performance index for this system is

$$J = \sum_{k=0}^{\infty} (x_k^T C^T C x_k + \rho^2 u_k R u_k) \tag{4}$$

Let us assume that the pair (C, A) is observable, the matrix CB has rank m, and that A is nonsingular. This last assumption on the nonsingularity of A is equivalent to the assumption that there are no pure time delays. It could be relaxed but leads to special cases requiring special treatment.

The optimal control is given by

$$u_k = Kx_k \tag{5}$$

with

$$K = -(\rho^2 R + B^T PB)^{-1} B^T PA \tag{6}$$

where P is the positive definite symmetric solution of

$$P = A^T PA + C^T C - (B^T PA)^T (\rho^2 R + B^T PB)^{-1} B^T PA \tag{7}$$

The return difference matrix is

$$T(z) = I - K(zI-A)^{-1}B \tag{8}$$

and the determinant of T(z) is

$$\det (T(Z)) = \frac{|zI - A - BK|}{|zI - A|} \quad \frac{\phi_c(z)}{\phi_o(z)} \tag{9}$$

where $\phi_c(z)$ is the closed-loop characteristic polynomial and $\phi_o(z)$ is the open-loop characteristic polynomial. Multiplying equation (7) on the left by $B^T (z^{-1}I - A^T)^{-1}$ and on the right by $(zI - A)^{-1}B$, and using equations (4) and (8), algebraic manipulation yields

$$\begin{aligned} &T^T(z^{-1}) (\rho^2 R + B^T PB) T(z) \\ &\quad = \rho^2 R + B^T (z^{-1}I - A^T)^{-1} C^T C (zI - A)^{-1} B \end{aligned} \tag{10}$$

It is possible to equate the determinants of both sides of equation (10) and examine the asymptotic nature of the eigenvalues of the closed-loop system as ρ tends to zero. However, in this discrete case the solution exists for $\rho = 0$ (namely, $P = C^T C$, and $K = -(CB)^T CA$, and the limiting behavior can be determined directly. Setting $\rho = 0$ and equating the determinants of the two sides of equation (10) yields

$$\frac{\phi_c(z^{-1})}{\phi_o(z^{-1})} \frac{\phi_c(z)}{\phi_o(z)} \det(B^T PB) = \det[H^T(z^{-1}) H(z)] \tag{11}$$

where

$$H(z) = C(zI-A)^{-1}B \tag{12}$$

It is convenient to introduce the following additional notation.

$$\phi_o(z) = a_o \prod_{i=1}^{n} (z-z_{io}), \quad \tilde{\phi}_o(z) = a_o \prod_{i=1}^{n} (1-zz_{io})$$

$$\phi_c(z) = a_c \prod_{i=1}^{n} (z-z_{ic}), \quad \tilde{\phi}_c(z) = a_c \prod_{i=1}^{n} (1-zz_{ic}).$$

$$\det H(z) = \left(\sum_{i=0}^{n-m} \rho_i z^i \right) / \phi_o(z), \quad \rho_{n-m} \neq 0$$

Then

$$\phi_o(z^{-1}) = z^{-n} \tilde{\phi}_o(z), \quad \phi_c(z^{-1}) = z^{-n} \tilde{\phi}_c(z),$$

and

$$\det H(z^{-1}) = z^m \sum_{i=0}^{n-m} \rho_i z^{n-m-i}) \;/\; \tilde{\phi}_o(z)$$

so that equation (11) may be written as

$$\phi_c(z)\, \tilde{\phi}_c(z) \det (B^T PB) = z^m \left(\sum_{i=1}^{n-m} \rho_i z^i \right) \left(\sum_{i-1}^{n-m} \rho_i z^{n-m-i} \right) \tag{13}$$

Thus there are m closed-loop eigenvalues at the origin and the remaining n-m closed-loop eigenvalues are at the zeros of the determinant of $H(z)$ or their inverses, whichever have magnitudes less than one.

The m eigenvectors x_i, associated with the zero eigenvalues are characterized by the equation

$$x_i = A^{-1} B \gamma_i, \; i = 1, 2, \ldots, m \tag{14}$$

where γ_i are m linearly independent m-dimensional vectors. If C is such that all the zeros, z_i, of $\det(H(z))$ have magnitude less than unity, then the corresponding eigenvectors are characterized by*

$$x_i = (z_i I - A)^{-1} B \gamma_i, \; i = 1, 2, \ldots, n-m \tag{15}$$

---

*Assuming z_i is not an eigenvalue of A.

where the m-dimensional vectors γ_i satisfy

$$C(z_i I - A)^{-1} B \gamma_i = 0 \tag{16}$$

Thus, a constructive procedure to approximately achieve desired modal properties for discrete systems consists of specifying the desired properties in terms of n-m desired eigenvalues z_{d_i} and the corresponding eigenvalues x_{d_i}. Then, find the closest achievable eigenvalues and eigenvectors. For example, take x_i to be the closest achievable vector to x_{d_i} corresponding to z_{d_i}--i.e.,*

$$x_i = (z_{d_i} I - A)^{-1} B \hat{\gamma}_i \tag{17}$$

where

$$\hat{\gamma}_i = \min \; || x_{d_i} - (z_{d_i} I - A)^{-1} B \gamma || \tag{18}$$

for i = 1, 2, ..., n-m and then choose C to be orthogonal to each x_i subject to the conditions that Rank C = CB = m, and (C, A) observable.

Of course, other methods of approximation may be used. Other special cases such as more than m zero eigenvalues, desired eigenvalues coinciding with open-loop eigenvalues, systems with pure time delays, etc., may be

---

*Assuming z_{d_i} is not an eigenvalue of A.

treated in a similar fashion, but the number of such cases and the special complexities introduced preclude their treatment here.

SECTION 3

LOOP TRANSMISSION PROPERTIES OF MULTIVARIABLE SYSTEMS

While LQ designs provide guaranteed stability margins, they require full state feedback and force the feedback loop to have a high frequency rolloff rate of only 1/s. In order to implement an LQ design, a Kalman filter or observer may be needed to reconstruct a state estimate from measured outputs. Furthermore, additional dynamic compensation may be required to provide adequate high-frequency attenuation. Each of these two forms of dynamics in the feedback path may significantly reduce the stability margins for the final design.

The standard LQG design employs a Kalman filter to construct an input using a quadratic optimal state feedback gain on the state estimate. In Appendix C we present an example of a legitimate LQG controller which has vanishingly small stability margins. Thus, the loop transmission properties of LQG controllers must be checked for each design and compensation either added or adjusted as appropriate.

In order to achieve desired feedback loop transmission properties we require:

1. A means of characterizing the loop transmission properties and specifying what is desirable.
2. Compensation structures which provide for loop shaping.

3. Methods for adjusting the feedback gains and/or Kalman filter.

4. The characterization of loop transmission properties leads to an alternative method for selecting quadratic weighting matrices.

These four aspects are treated in order below.

CHARACTERIZATION OF LOOP TRANSMISSION PROPERTIES

Consider the linear, time-invariant feedback system in Figure 1 where for simplicity the plant and all compensation have been lumped into the transfer function Q(s). We will be concentrating on the feedback properties of this system where Q(s) may, in general, be a transfer function matrix.

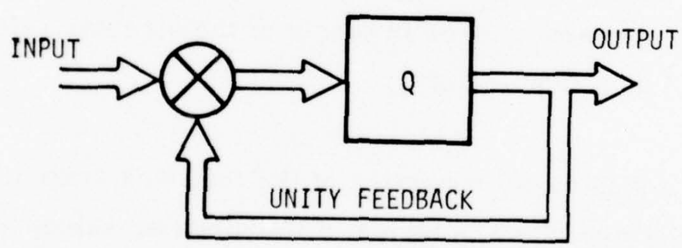


Figure 1. Feedback System

The critical feedback issue is to provide desired performance in the face of plant uncertainty (see Appendix D; also, [4]). The effect of feedback on a system's uncertainty is related to its loop transmission properties, which for single input/output systems are most naturally and usefully characterized in terms of the frequency response of the loop transfer and return difference functions $(Q(s),\ 1 + Q(s),\ 1 + 1/Q(s))$. This is often represented in various graphical forms such as Bode plots, Nyquist and Inverse Nyquist Diagrams, and Nichols charts.

For multiple input/output systems, the analogous characterization of the loop transmission properties is in terms of the singular values and vectors of the corresponding feedback matrices $(Q(s),\ I + Q(s),\ I + Q(s)^{-1})$. Appendix E develops this concept and demonstrates the applicability of singular value analysis to issues of robustness for multivariable systems. The major results are:

- The gain and rotation properties of a transfer function matrix $Q(s)$ may be characterized in terms of the singular values and vectors of $Q(j\omega)$ as a function of ω.

- The robustness properties of the feedback system in Figure 2 may be characterized in terms of the singular values and vectors of $I + Q^{-1}(s)$ and $I + Q(s)$. In particular, for the system in the figure where $L(s)$ represents a stable perturbation matrix of a nominally stable feedback system, the perturbed system will remain stable if

$$\underline{\sigma}\,(I + Q^{-1}(j\omega)) > \bar{\sigma}\,(L\,(j\omega)) \quad \text{for all } \omega$$

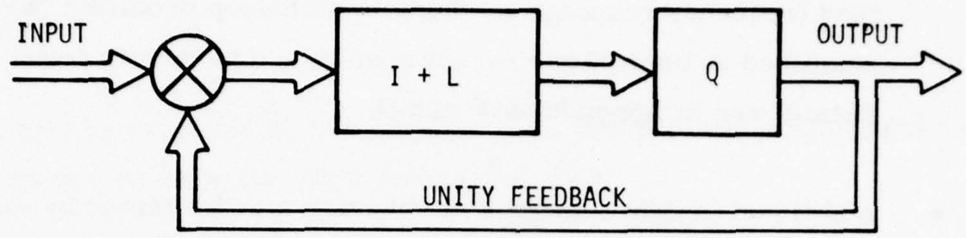


Figure 2. Perturbed Feedback System

where $\underline{\sigma}$ and $\bar{\sigma}$ denote minimum and maximum singular values, respectively. This gives a measure of system robustness in terms of $\bar{\sigma}(I + Q^{-1}(j\omega))$ which is analogous to the distance from the critical point on an Inverse Nyquist Diagram.

The above bound is tight in the sense that for each frequency a destabilizing L exists with $\bar{\sigma}(L) \geq \underline{\sigma}(I + Q^{-1})$. Such a destabilizing L may not necessarily be rational or even causal.

- Some alternative approaches for evaluating multivariable stability margins have been shown to be inadequate. In particular, loop breaking, Characteristic Loci and methods based on diagonalization give unreliable information on robustness.

- Two somewhat popular control design methodologies which use the Inverse Nyquist Array and Characteristic Loci in an attempt to extend frequency response methods to multiloop problems have been examined in light of our recent results and have been found wanting. Details are in Appendixes F and G.

- Additional insight into system behavior may be gained by examining singular values of matrices as a function of s and not just $j\omega$. This leads to mappings defined on multiple copies of the complex plane forming Riemann surfaces. See Appendix H for details.

- The robustness aspects of the singular value approach fits neatly into the more general stability theory developed by Safonov [5]. Determination of the implications of this will require further research.

LOOP SHAPING BY DYNAMIC COMPENSATION

In order to achieve desirable loop transmission properties, it is often necessary to add additional dynamic compensation in the feedback path. This compensation may significantly alter not only the loop transmission properties, but also the closed-loop behavior of the system. Since we have characterized the plant performance in terms of modal properties, it would be desirable to find compensation structures which preserve the closed-loop plant eigenvalues and eigenvectors associated with the original full state feedback design. Since it is well known that use of an observer or filter to reconstruct the input signal preserves closed-loop plant modal

characteristics, mode preserving compensation structures would provide a complete methodology for implementation of feedback designs based on modal properties.

We will begin by considering the compensation problem for single input systems. Suppose we have the state feedback system

$$\dot{x} = Ax + bu \tag{19}$$

$$u = f_1 x \tag{20}$$

where $x \in R^n$. Then

$$f_1(sI - A)^{-1}b = \frac{n_1(s)}{d_o(s)} \tag{21}$$

and

$$1 + f_1(sI - A)^{-1}b = \frac{n_1(s) + d_o(s)}{d_o(s)} = \frac{d_c(s)}{d_o(s)} \tag{22}$$

where

$$d_o(s) = \det(sI - A) = s^n + a_1 s^{n-1} + \dots a_{n-1}s + a_n \tag{23}$$

is the open-loop characteristic polynomial and

$$d_c(s) = \det(sI - A + bf_1) = s^n + c_1 s^{n-1} + \dots c_{n-1}s + c_n \tag{24}$$

is the closed-loop characteristic polynomial.

In order to add compensation without changing the closed loop plant eigenvalues we must find $d_p(s)$ and f_2 such that

$$1 + \frac{1}{d_p(s)} f_2(sI - A)^{-1}b$$

$$= \frac{n_2(s) + d_o(s)d_p(s)}{d_o(s)d_p(s)}$$

$$= \frac{d_c(s)d_r(s)}{d_o(s)d_p(s)} \tag{25}$$

where $n_2(s) = d_o(s)f_2(sI - A)^{-1}b$ is a polynomial of order no greater than $n - 1$.

Let

$$d_p(s) = s^m + g_1 s^{m-1} + \ldots g_{m-1}s + g_m \tag{26}$$

$$d_r(s) = s^m + h_1 s^{m-1} + \ldots h_{m-1}s + h_m \tag{27}$$

Then

$$n_2(s) = d_c(s)d_r(s) - d_o(s)d_p(s) \tag{28}$$

implies that the first $n-m$ coefficients of $d_c(s)d_r(s) - d_o(s)d_p(s)$ be zero. This in turn implies that

$$\left.\begin{aligned} h_1 &= g_1 + (c_1 - a_1) \\ h_2 &= g_2 + (c_2 - a_2) + (c_1 - a_1)(g_1 - a_1) \\ h_3 &= g_3 + (c_3 - a_3) + (c_2 - a_2)(g_1 - a_1) + \\ &\qquad (c_1 - a_1)(g_2 - a_2 - a_1^2 - a_1 g_1) \end{aligned}\right\} \tag{29}$$

etc.

By thus restricting $d_p(s)$ and $d_r(s)$ we guarantee that we can find a state feedback f_2 such that equation (25) is satisfied. *This allows us to concentrate* on the adjustment of the parameters in the compensator while maintaining the desired closed-loop plant modal properties.

As a simple example consider a system with

$$d_o(s) = s^2 - 2s + 26 \tag{30}$$

with open loop poles at $1 \pm 5j$ and

$$d_c(s) = s^2 + 10s + 50 \tag{31}$$

with closed loop poles at $-5 \pm 5j$. We will consider several cases:

A. no compensation,

with
$$d_r(s) = s + g_1, \tag{32}$$

$$d_p(s) = s + g_1 + 12, \tag{33}$$

B. $g_1 = 10$,

C. $g_1 = 6$,

D. $g_1 = 4$,

and with
$$d_r(s) = s^2 + g_1 s + g_2, \tag{34}$$

$$d_p(s) = s^2 + (g_1 + 12)s + g_2 + 12g_1 + 48, \tag{35}$$

E. $g_1 = 20$, $g_2 = 100$ poles $(-10, -10)$

F. $g_1 = 12$, $g_2 = 70$ poles $(-6 \pm 5.6j)$

The Nyquist diagrams are shown for cases A, B, and D in Figure 3 and for cases C, E, and F in Figure 4. The loop gain Bode plots are shown in Figures 5 and 6. Note that, as expected, bandwidth is reduced at the expense of gain and phase margin. The parameters may be adjusted until an adequate trade-off is achieved.

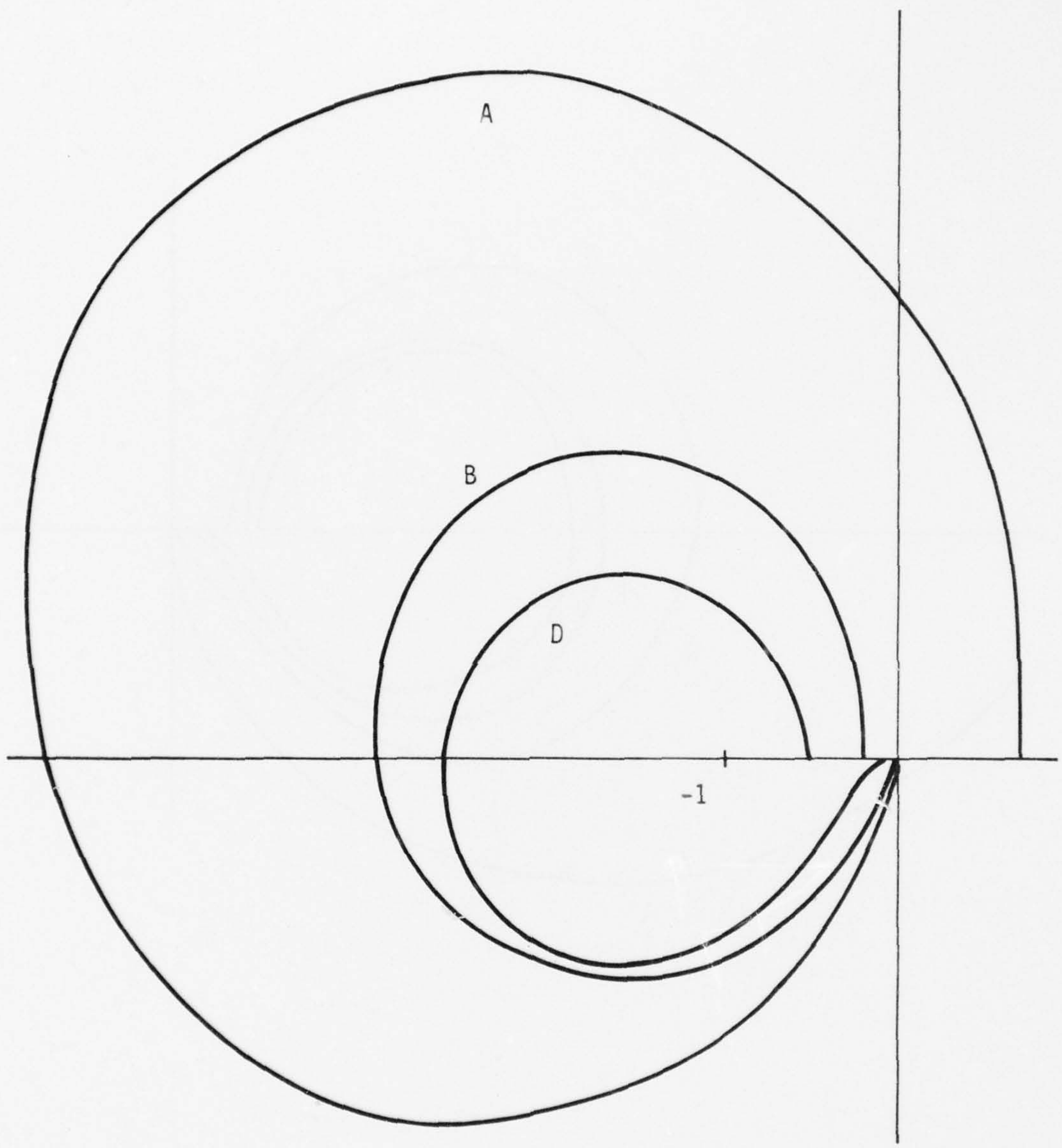


Figure 3. Nyquist Diagrams for Cases A, B, and D

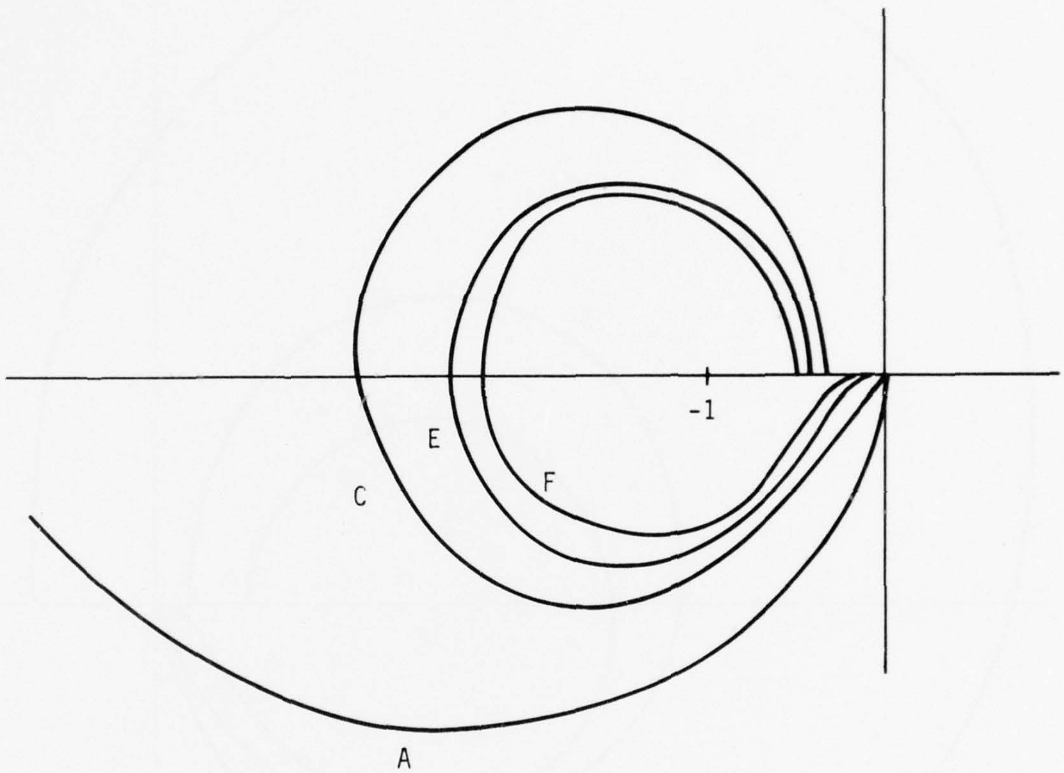


Figure 4. Nyquist Diagrams for Cases C, E, and F

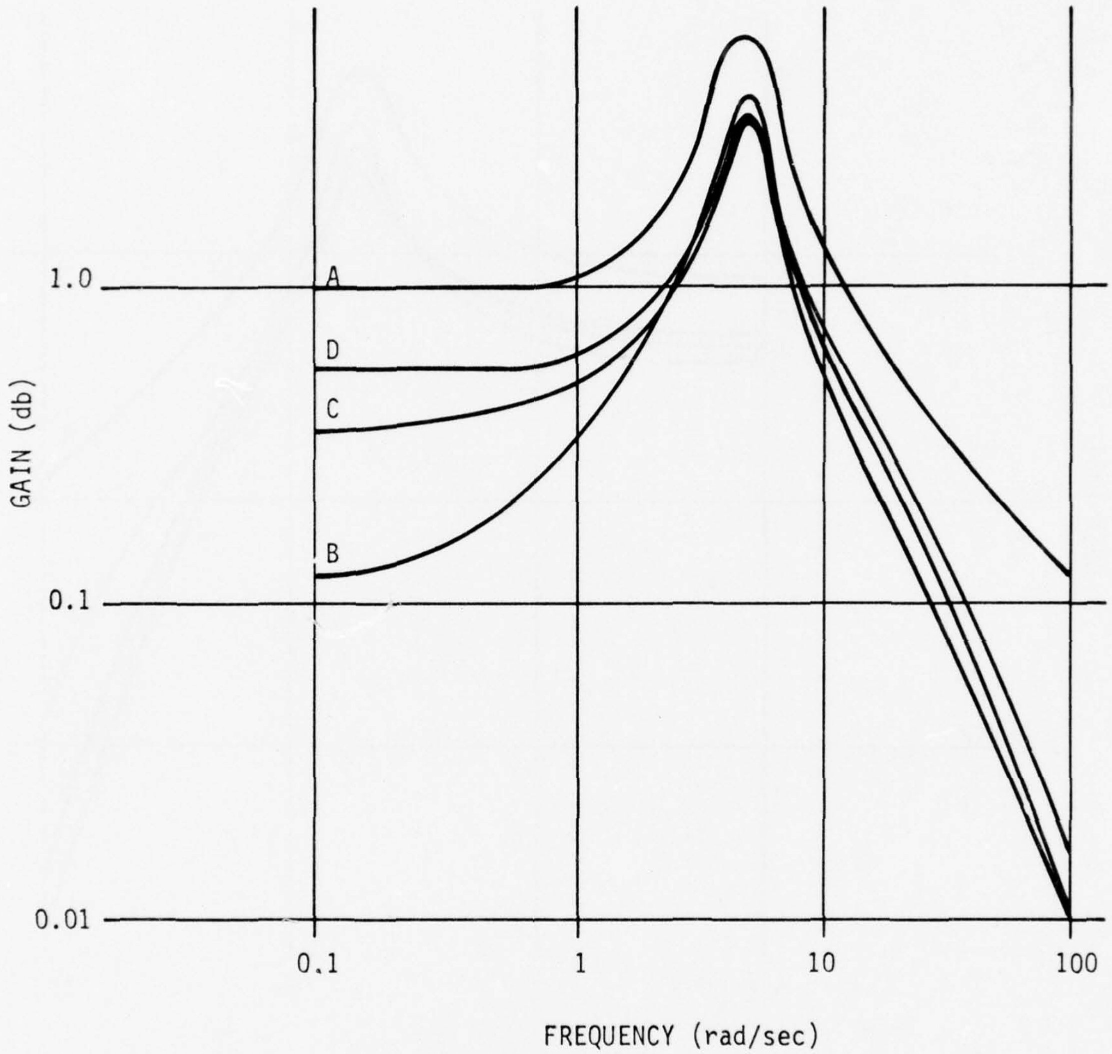


Figure 5. Bode Plots for Cases A, B, C, and D

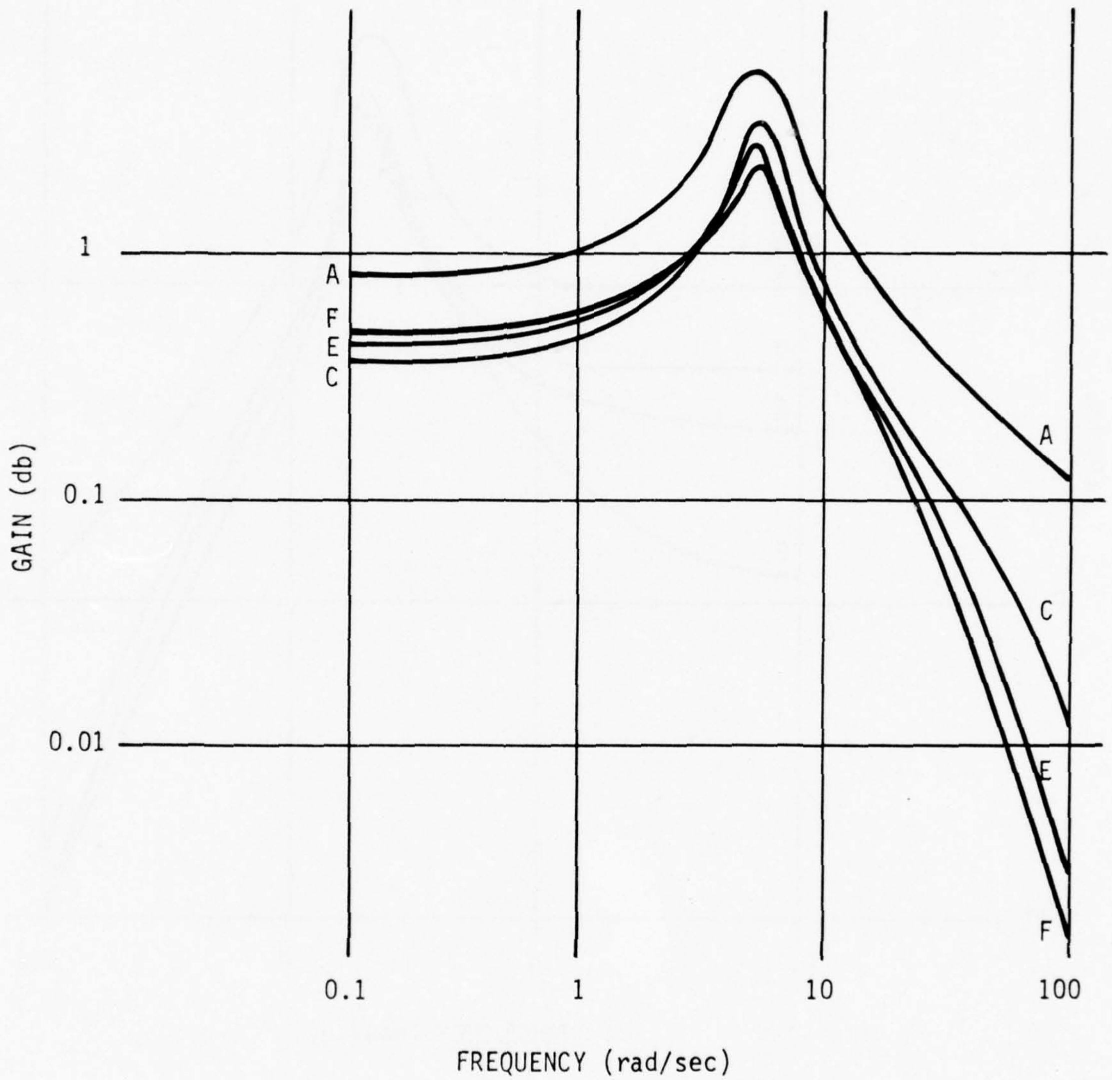


Figure 6. Bode Plots for Cases A, C, E, and F

The compensation problem becomes substantially more complicated for multiple input problems.

Suppose we have a state feedback design

$$\dot{x} = A_1 x + Bu \tag{36}$$

$$u = -Fx \tag{37}$$

where the loop transfer function is

$$F(sI - A_1)^{-1}B \tag{38}$$

This design is assumed to have desirable properties in terms of the closed-loop eigenvalues and eigenvectors.

These may be related directly to the feedback gains [6]. Let s_i and v_i denote the ith closed-loop eigenvalue and eigenvector, respectively. Then there exists a $w_i \epsilon C^m$ such that

$$(s_i I - A_i)v_i = -Bw_i \tag{39}$$

for each i, $i = 1, \ldots, n$. Then

$$F = WV^{-1} \tag{40}$$

where $W = [w_1, \ldots, w_n]$ and $V = [v_1, \ldots, v_n]$. Also

$$VS - AV = BW \tag{41}$$

where $S = \text{diag}\ [s_1, \ldots, s_n]$. Two types of compensation may be used:

1. <u>Compensation on Inputs</u>

Let $-(sI - A_2)^{-1}A_2\hat{F}(sI - A_1)^{-1}B$ be the compensated loop transfer function and $\hat{F}$ the new feedback matrix. In order that the closed-loop eigenvalues and eigenvectors of the plant remain the same it is necessary that

$$\begin{aligned} v_i &= (s_iI - A_1)^{-1}B(s_iI - A_2)^{-1}A_2\hat{w}_i \\ &= (s_iI - A_1)^{-1}Bw_i \end{aligned} \tag{42}$$

This implies that

$$w_i = (sI - A_2)^{-1}A_2\hat{w}_i \tag{43}$$

or

$$\hat{W} = A_2^{-1}WS - W, \tag{44}$$

with

$$\hat{F} = \hat{W}V^{-1} \tag{45}$$

Then the compensated loop transfer matrix has the form

$$-(sI - A_2)^{-1} A_2 (W - A_2^{-1} WS) V^{-1} (sI - A_1)^{-1} B$$

$$= -(sI - A_2)^{-1} (A_2 W - WS) V^{-1} (sI - A_1)^{-1} B \tag{46}$$

with closed-loop plant eigenvalues $\{s_i\}$ and eigenvectors $\{v_i\}$. The A_2 may be selected to shape the transfer function while the eigensystem of the plot remains fixed.

Note that the closed-loop system has a state matrix

$$\begin{bmatrix} A_1 & B \\ -B\hat{F} & A_2 \end{bmatrix}$$

and

$$\mathrm{Tr} \begin{bmatrix} A_1 & B \\ -B\hat{F} & A_2 \end{bmatrix} = \mathrm{Tr} \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix} \tag{47}$$

Since the trace of a matrix is equal to the sum of its eigenvalues, the sums of the open and closed loop eigenvalues are the same. Thus, there is one limitation on how much lag may be introduced before instability occurs.

This method adapts easily to allow for feedback designs which do not include actuator position feedback--i. e., A_2 could be associated with actuator states.

2. Compensation on States

In a similar manner, compensation on the states may be used to give the loop transfer matrix

$$-\bar{F}(sI - A_3)^{-1}A_3(sI - A_1)^{-1}B.$$

It is easily shown that to preserve the original closed loop eigensystem

$$\bar{F} = W\bar{V}^{-1} \tag{48}$$

where

$$A_3\bar{V} - \bar{V}S = A_3V. \tag{49}$$

These two methods may be combined to yield a loop transfer function of

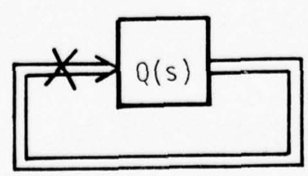
$$(sI - A_2)^{-1}(A_2W - WS)\bar{V}^{-1}(sI - A_3)^{-1}A_3(sI - A)^{-1}B$$

The procedure for selecting the A_2 and A_3 is fairly ad hoc and amounts to trial and error shaping of the singular values. In practice, A_2 and A_3 will generally be selected as diagonal, and simultaneous loop shaping may be done by adjusting the diagonal elements. The procedure outlined above simply provides a structure that preserves the closed-loop eigensystem. All of this depends, of course, on the somewhat dubious assumption that the closed-loop eigensystem adequately characterizes the plant performance. Further research will be required before an entirely adequate characterization of system performance is obtained.

ADJUSTMENT OF OBSERVER/FILTER DESIGN TO ACHIEVE ROBUSTNESS

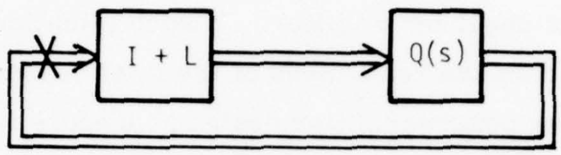
Although LQ state feedback systems have guaranteed stability margins, no such guarantee exists for the full LQG system with Kalman filter, as the example in Appendix C shows. More generally the robustness properties of any state feedback design may be seriously degraded when that design is implemented with an observer or filter. In such situations, a systematic methodology to improve the robustness of the final design would be desirable. We have developed an observer/filter design procedure which accomplishes this. To provide insight into the nature of the trade-offs involved, we present the results in terms of a particular equivalence between noise response and robustness.

Two properties which are important in feedback systems, robustness with respect to perturbations and noise rejection, may be viewed as two aspects of a single, more fundamental property. To see this, consider a closed-loop LTI feedback system where some set of p points in signal paths have been identified. This set of points will be denoted by X. These may be anywhere in the system such as in feedback loops or internal signal paths where perturbations occur, or places where noise enters. Suppose the system has been rearranged to isolate the selected signal paths as shown:

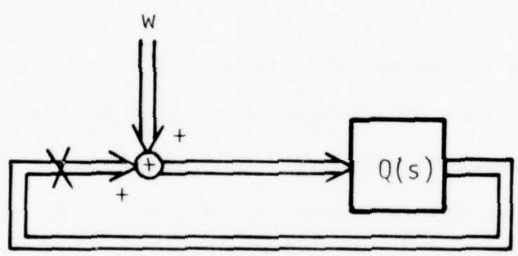


Here Q(s) represents the Laplace transform function of the whole system if the loops were broken at the selected points, X.

Now consider two possible types of uncertainty introduced at X, a perturbation L (nominally zero)



of bounded spectral norm (as a function of frequency) and white noise w



with intensity I. (See Appendix E for more details on perturbations of the type indicated.)

It has been shown (Appendix E) that the system will remain stable for arbitrary perturbations L provided

$$\underline{\sigma}\ (I - Q^{-1}(j\omega)) > \|L\|_2 = \bar{\sigma}\ (L) \tag{50}$$

where $\underline{\sigma}$, $\overline{\sigma}$, and $||\cdot||_2$ denote the minimum and maximum singular values and spectral norm, respectively. Thus $\underline{\sigma}(I - Q^{-1})$ characterizes robustness of Q to perturbations L at X.

Furthermore, the response at X of Q to noise w has a spectrum in terms of $[I - Q^{-1}(j\omega)]^{-1}$, and the maximum response is

$$\overline{\sigma}([I - Q^{-1}(j\omega)]^{-1}) = \frac{1}{\underline{\sigma}(I - Q^{-1}(j\omega))} \tag{51}$$

in the singular vector directions associated with $\underline{\sigma}$. Thus $\underline{\sigma}(I - Q^{-1})$ characterizes the response to noise w at X.

It can now be clearly seen that both robustness and noise rejection depend on $\underline{\sigma}(I - Q^{-1})$ (or more generally $I - Q^{-1}$), and both properties are improved when this quantity is large. In particular, two systems which have identical response to noise at some point X will have identical robustness with respect to parameter variations at X, and vice-versa.

By exploiting this equivalence between noise response and robustness it is possible to design Kalman filters strictly on the basis of noise response which provide desired robustness properties when used for feedback. This is easily seen by considering the state feedback system

$$\dot{x} = Ax + Bu + Gw \tag{52}$$

$$y = Cx + v \tag{53}$$

$$u = -Fx \tag{54}$$

where $x \in R^n$ is the state, $u \in R^p$ is process noise with $E(w(t)\, w(\tau)^T) = W\,\delta(t-\tau)$, $y \in R^r$ is the output, and $v \in R^r$ is the observation noise with $E(v(t)\, v(\tau)^T) = 1/q\; V\,\delta(t-\tau)$. The closed-loop response of x to the noise **w** is

$$x = (sI - A + BF)^{-1}\, Gw. \tag{55}$$

We assume that the transfer function $C(sI - A + BF)^{-1}G$ has no right-half plane zeroes.

Suppose the feedback design is to be implemented using a Kalman filter to generate a state estimate $\hat{x}$ where

$$\dot{\hat{x}} = (A - KC - BF)\,\hat{x} + Ky \tag{56}$$

$$u = -F\hat{x}. \tag{57}$$

with K being the Kalman filter gain. Letting $e = x - \hat{x}$ we can write the response of the state as

$$x = (sI - A + BF)^{-1}\,[Gw + BFe] \tag{58}$$

where

$$e = (sI - A + KC)^{-1}\,[Gw - Kv]. \tag{59}$$

Suppose $r = p$ and let the observation noise intensity $1/q^2\ V$ go to zero by letting $q \to \infty$. Then asymptotically

$$\lim_{q \to \infty} K \to q\, G\, W_1\, V_1^{-1} \tag{60}$$

where W_1 denotes some square root of W ($W_1 W_1^T = W$) and, similarly, V_1 is some square root of V. Then

$$\begin{aligned} e &= (sI - A + KC)^{-1}\,[Gw - Kv] \\ &\to (sI - A + qGW_1V_1^{-1}C)^{-1}[G - qGw_1V_1^{-1}v] \\ &= (sI - A)^{-1}\, G[I + qW_1V_1^{-1}C(sI - A)^{-1}G]^{-1}(w - qW_1V_1^{-1}v) \\ &\to (sI - A)^{-1}\, G[W_1V_1^{-1}C(sI - A)^{-1}\, G]^{-1}(\tfrac{1}{q}\, w - W_1V_1^{-1}v) \end{aligned} \tag{61}$$

Hence, in the limit, $E\{e(t)e(\tau)^T\}\ 0$ since $E\{w(t)v(\tau)\} = 0$ and

$E\{\frac{1}{q}\, w(t)[\frac{1}{q}\, w(\tau)]^T\} = E\{W_1V_1^{-1}v(t)[W_1\, V_1^{-1}v(\tau)]^T\}$ for all $q \geq 0$.

Thus, with the Kalman filter in the loop, as $q \to \infty$, the noise response of the state asymptotically approaches that for the full state feedback design. That is,

$$x \to (sI - A + BF)^{-1}\, Gw. \tag{62}$$

Because of the previously noted equivalence between noise response and robustness, this implies that at the point where the noise is introduced, the

robustness properties of the design using the Kalman filter asymptotically approaches the robustness properties of the state feedback design. The number of points where the perturbations or noise may be introduced is equal to the number of independent measurements available. This asymptotic property is achieved at the expense of poor rejection of observation noise, and only holds for systems where the transfer function matrix between the fictitious noise processes and the measurements has no right-half plane zeroes. The trade-off between observation noise response and achievement of state feedback properties may be obtained through adjustment of the parameter q.

This approach may be used to design robust controllers using a modified LQG methodology. Here fictitious noise may be added in the Kalman filter design step to improve the robustness properties of the final design. Optimal noise rejection may then be traded off with robustness. Appendix I presents an example of such an approach.

This methodology applies equally well to observer design. Any stable observer whose gain behaves as in equation (60) will have the desired properties. The Kalman filter framework provides a convenient approach to observer design.

AN ALTERNATE METHOD FOR SELECTING QUADRATIC WEIGHTS BASED ON SINGULAR VALUES

Consider the LQ problem for the system

$$\dot{x} = Ax + Bu, \quad x(0) \text{ given} \tag{63}$$

with performance index

$$J = \int_0^{\infty} [\,(Cx)^T Cx + u^T u]\, dt \tag{64}$$

where x is an n-vector and u is an m-vector. The optimal control is

$$u = Fx \tag{65}$$

with

$$F = -B^T P \tag{66}$$

where P is the positive definite symmetric solution of

$$C^T C = PBB^T P - PA - A^T P \tag{67}$$

This is equivalent to F being the solution of

$$[I + G(j\omega)]^* [I + G(j\omega)] = I + H^*(j\omega)H(j\omega) \tag{68}$$

where $G(j\omega) = F(j\omega I - A)^{-1}B$ (69)

and $H(j\omega) = C(j\omega I - A)^{-1}B$ (70)

In terms of singular values, equation (68) yields

$$\sigma^2(I + G(j\omega)) = 1 + \sigma^2(H(j\omega)) \tag{71}$$

This gives a characterization of the frequency domain properties of $(I + G)$ directly in terms of the quadratic weighting matrix $C^T C$.

If one is given desired singular values of $(I + G(j\omega)$, then these define desired singular values of $H(j)$. Suppose that

$$H_d(j\omega) = \hat{U}\hat{\Sigma}\hat{V}^* \tag{72}$$

possesses the desired singular properties, and let the singular values decomposition of $(j\omega I - A)^{-1}B$ be

$$(j\omega I - A)^{-1}B = U\Sigma V^* \tag{73}$$

Then the weighting matrix C may be chosen such that $CU\Sigma V^*$ approximates $H_d(j\omega)$ in some manner. For example, select C to minimize

$$||CU\Sigma V^* - \hat{U}\hat{\Sigma}\hat{V}^*||_Q \tag{74}$$

where all terms except C in (74) are functions of frequency ω. With C so chosen, the Riccati equation (67) may be solved and the gain matrix obtained from equation (66). The choice of the norm to be used in (74) that yields good closed-loop properties requires further investigation.

Another area that appears to be worthy of future research is based on the following observation. Suppose that in equation (63), A is stable with the control given by (65), (66), and (67) where C is such that

$$C^T C = PBB^T P - Q \tag{75}$$

with

$$Q \geq 0 \tag{76}$$

This is equivalent to the condition that

$$PA + A^TP \leq 0 \tag{77}$$

In this case the controlled system has what is called integrity. That is, if the gain of any actuator or set of actuators is reduced or even becomes zero, the remaining system will have guaranteed robustness of LQ controllers. This result may be established as follows. Let D be a diagonal matrix with elements d_i satisfying

$$0 \leq d_i \leq 1 \tag{78}$$

and set

$$\hat{B} = BD \tag{79}$$

Then the control is

$$u = \hat{F}x \tag{80}$$

with

$$\hat{F} = -\hat{B}^TP = DF \tag{81}$$

This control is optimal with respect to (64) with C replaced by $\hat{C}$ where

$$\hat{C}^T\hat{C} = PBDD^TB^TP - Q \tag{82}$$

This result has important implications for robustness of feedback systems for open-loop systems that are stable, and further research is recommended to deal with open-loop systems that are unstable.

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APPENDIX A

CHOOSING QUADRATIC WEIGHTS TO ACHIEVE DESIRED ASYMPTOTIC MODAL PROPERTIES

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CHOOSING QUADRATIC WEIGHTS TO ACHIEVE DESIRED ASYMPTOTIC MODAL PROPERTIES

By H. R. Sirisena

ABSTRACT

A new approach to the characterization of quadratic performance indices in terms of asymptotic eigenvalues and eigenvectors is described. The regular case where precisely n-m eigenvalues remain finite for an n-th order m-input system is first treated. It is shown that there exists an m-1 parameter family of quadratic performance indices that yields any desired set of asymptotic eigenvalues and eigenvectors. A solution is also obtained for the two leading terms in the asymptotic expansion of the optimal control law.

The singular case where more than n-m asymptotic eigenvalues remain finite is treated next. It turns out that in this case there is less freedom in specifying the asymptotic modes and moreover the corresponding quadratic performance indices are characterized by nonlinear equations rather than by linear equations as in the regular case. Hence solutions may not exist in general, and in any case are difficult to obtain.

1. Introduction

Consider the n-th order, m-input plant

$$\dot{x} = Fx + Gu \tag{1}$$

associated with the quadratic performance index

$$J = \int_0^\infty (x'Qx + \rho^2 u' Ru)dt \tag{2}$$

where, without loss of generality [1], it is assumed that $Q = H'H$, H being an mxn matrix. A study of the asymptotic modal properties of the optimal closed-loop system as $\rho \to 0$ [1], [2] reveals that three cases can be distinguished.

(i) When rank (HG) = m, n-m eigenvalues remain finite while the remaining m eigenvalues tend to infinity in m first-order Butterworth patterns.

(ii) When rank H = rank G = m, but rank (HG) < m, fewer than n-m eigenvalues remain finite while the remaining eigenvalues tend to infinity in multiple Butterworth patterns

(iii) When rank H < m, more than n-m eigenvalues may remain finite.

Here we consider the inverse problem of finding weighting matrices Q and R such that the optimal system possesses prespecified modal properties. The inverse problem corresponding to case (i) has already been studied in reference [1], but here we give an alternative treatment that also yields the two leading terms in the asymptotic expansion of the optimal control law. The approach adopted also enables investigation of the inverse problem corresponding to case (iii).

2. The Case Rank (HG) = m

Optimal Control Law

The optimal state feedback law K is given by

$$K = -\rho^{-2}R^{-1}G'P \tag{3}$$

Where P is the symmetric positive-definite solution of the algebraic Riccati equation (ARE)

$$PF + F'P - \rho^{-2}PGR^{-1}G'P + Q = 0 \tag{4}$$

As $\rho \to 0$, P can be expanded in a power series

$$P = \rho P_1 + \rho^2 P_2 + \rho^3 P_3 + \dots \quad (5)$$

where the matrices P_i are all symmetric, and hence

$$K = \rho^{-1}K_1 + K_2 + \rho K_3 + \dots \quad (6)$$

where

$$K_i = - R^{-1}G'P_i \quad (7)$$

It is clear that the asymptotic properties of the optimal system as $\rho \to 0$ are governed by the first two terms in the expansion (6).

Asymptotically Finite Eigenvalues

Let (s_i, x_i), i = 1,2..., n-m be the desired finite asymptotic eigenvalues (assumed to be distinct), and the corresponding eigenvectors respectively. Then we have that

$$\lim_{\rho \to 0} (s_i I-F-GK)x_i = 0, \quad i = 1,2,\dots,n-m \quad (8)$$

Plugging in (6), it follows that

$$\lim_{\rho \to 0} (s_i I-F- \rho^{-1} GK_1 - GK_2)x_i = 0, \ i = 1,2,\dots,n-m \quad (9)$$

Now from consideration of the $O(\rho^{-1})$ term, it is evident that

$$GK_1 x_i = 0, \quad i = 1,2,\dots,n-m$$

which implies that

$$K_1 x_i = 0 \qquad i = 1,2,\dots,n-m \quad (10)$$

providing G has full rank m.

From (9) and (10), we also have that

$$(s_i I-F-GK_2)x_i = 0, \ i = 1,2,\dots,n-m \quad (11)$$

which, following Moore [3], can be rearranged into the form

$$x_i = (s_i I-F)^{-1} GK_2 x_i \stackrel{\Delta}{=} (s_i I-F)^{-1} G \nu_i, \ i = 1,2,\dots,n-m \quad (12)$$

where

$$\nu_i = K_2 x_i, \ i = 1,2,\dots,n-m \quad (13)$$

(12) implies that once s_i is specified, the corresponding eigenvector x_i cannot be specified arbitrarily but is confined to an m-dimensional subspace. The elements of ν_i provide m-1 independent parameters for the specification of x_i.

Asymptotically Infinite Eigenvalues

From (9) it is clear that the m eigenvalues that tend to infinity as $\rho \to 0$ must be of the form $\rho^{-1} s_i^\infty$, $i = 1,2,\ldots,m$, where the s_i^∞ are finite. Moreover,

$$(\rho^{-1} s_i^\infty I - \rho^{-1} GK_1) x_i^\infty = 0, \; i = 1,2,\ldots,m \tag{14}$$

i.e., the s_i^∞ and x_i^∞ are eigenvalue-eigenvector pairs of the matrix GK_1. We shall assume that the s_i^∞ are non-zero and distinct.

(14) can also be expressed as the pair of equations

$$\left.\begin{aligned} x_i^\infty &= G\nu_i^\infty \\ K_1 x_i^\infty &= s_i^\infty \nu_i^\infty \end{aligned}\right\} \quad i = 1.2,\ldots,m \tag{15}$$

(15) implies that the asymptotically infinite modes x_i^∞ are confined to the m-dimensional range space of G, and are characterized by m-1 parameters (elements of ν_i^∞).

Evaluation of the Control Weighting Matrix R

Defining the matrices

$$X_\infty = \begin{bmatrix} x_1^\infty & x_2^\infty & \ldots & x_m^\infty \end{bmatrix}$$

$$N = \begin{bmatrix} \nu_1^\infty & \nu_2^\infty & \ldots & \nu_m^\infty \end{bmatrix}$$

$$S = \text{diag} \begin{bmatrix} s_1^\infty & s_2^\infty & \ldots & s_m^\infty \end{bmatrix},$$

(15) can be written concisely as

$$X_\infty = GN \tag{16}$$

$$K_1 X_\infty = NS \tag{17}$$

and we also have that

$$K_1 G = NSN^{-1} \tag{18}$$

Now $RK_iG = G'P_iG$ must be symmetric for every i. Hence

$$RK_1G = RNSN^{-1} = (RNSN^{-1})'$$

i.e., $\quad RNSN^{-1} = (N^{-1})'SN'R$

whence

$$N'RNS = SN'RN$$

which in turn implies that

$$N'RN = \Gamma \tag{19}$$

where Γ is any mxm matrix that commutes with S,

i.e. $\Gamma S = S \Gamma$

For the case in hand where S has distinct non-zero diagonal elements, Γ must also be diagonal though otherwise arbitrary. Hence from (19),

$$R = (N')^{-1} \quad \Gamma N^{-1} \tag{20}$$

where Γ is any arbitrary diagonal matrix. Since we require R to be positive definite, the diagonal elements of Γ must be strictly positive.

Evaluation of the Matrices $Q_1 . K_1$ and K_2

Using (3), the ARE (4) can also be written as

$$PF + F'P - \rho K'RK + Q = 0 \tag{21}$$

Substituting the power series expansions of P and K into (21) and equating O(1), O(ρ) and O(ρ^2) terms on the LHS to zero yields the three equations

$$Q - K_1'RK_1 = 0 \tag{22}$$

$$P_1F + F'Q_1 - K_1'RK_2 - K_2'RK_1 = 0 \tag{23}$$

$$P_2F + F'P_2 - K_1'RK_3 - K_2'RK_2 - K_3'RK_1 = 0 \tag{24}$$

Now defining the matrices

$$X_f = \left[x_1 \quad x_2 \cdots x_{n-m} \right]$$

$$N_f = \left[\nu_1 \quad \nu_2 \cdots \nu_{n-m} \right]$$

$$X = \left[X_f \quad X_\infty \right]$$

(10) and (13) may be written as

$$K_1 X_f = 0 \tag{25}$$

$$K_2 X_f = N_f \tag{26}$$

Then from (17) and (25) we obtain that

$$K_1 \left[X_f \quad X_\infty \right] = \left[0 \quad NS \right]$$

whence

$$K_1 = \left[0 \quad NS \right] \left[X_f \quad X_\infty \right]^{-1}$$

Defining

$$\left[X_f \quad X_\infty \right]^{-1} = \begin{bmatrix} Y_f \\ Y_\infty \end{bmatrix} \begin{array}{l} \updownarrow n-m \\ \updownarrow m \end{array}$$

it follows that

$$K_1 = NSY_\infty \tag{28}$$

Clearly the matrices Y_f and Y_∞ are comprised of the dual eigenvectors associated, respectively, with the asymptotically finite and asymptotically infinite eigenvalues.

Substituting for K_1 and R in (22), we obtain

$$Q = Y_\infty{}' \, S \Gamma S Y_\infty \tag{29}$$

It only remains to find K_2. Now K_2 is already partially specified by equation (26). To complete the characterization of K_2, multiply (23) on the left by G' and on the right by G to obtain

$$G'P_1FG + G'F'P_1G - G'K_1{}'RK_2G - G'K_2{}'RK_1G = 0$$

Substituting for P_1 in terms of K_1 and also invoking the symmetry of RK_2G, we have that

$$G'K_1{}' \, (RK_2G) + (RK_2G) \, K_1G + RK_1FG + G'F'K_1{}'R = 0 \tag{30}$$

Since K_1G is nonsingular, this Lyapunov equation has a unique solution for RK_2G, say

$$RK_2G = Z \tag{31}$$

From (31) and (16), we obtain

$$K_2GN = K_2X_\infty = R^{-1}ZN \tag{32}$$

and finally from (26) and (32) we find that

$$\begin{aligned} K_2 &= \begin{bmatrix} N_f & R^{-1}ZN \end{bmatrix} \begin{bmatrix} X_f & X_\infty \end{bmatrix}^{-1} \\ &= N_fY_f + R^{-1}ZNY_\infty \end{aligned} \tag{33}$$

<u>Discussion</u>

In view of the arbitrariness of the diagonal matrix Γ, (20) and (29) do not uniquely define the weighting matrices R and Q that yield optimal systems having the specified asymptotic modal properties. We also note that although K_1 is independent of Γ, K_2 appears to depend on the choice of Γ. Hence while every choice of Γ results in the same asymptotic properties, the properties of the optimal system for <u>finite</u> ρ do depend on the choice of Γ. Thus there remains

the possibility of choosing the m-1 independent parameters in Γ so as to meet subsidiary design objectives while preserving the desired asymptotic modal properties.

It is interesting to note that in view of equations (9) - (11), the feedback law

$$u = (\rho^{-1}K_1 + K_2)x \tag{34}$$

results in the (n-m) eigenvalue - eigenvector pairs (S_i, x_i), $i = 1,2,\ldots, n\text{-}m$ being invariant with respect to ρ. Thus the 'truncated' control law (29), which is asymptotically equivalent to the true optimal control law K_x, appears to possess properties that may be practically useful.

3. The Case Rank H < m

Suppose we want to have more than n-m asymptotically finite eigenvalues, say, s_i, $i = 1,2,\ldots,n\text{-}q$ where $q < m$. Suppose also that we want the remaining q eigenvalues to tend to infinity in first-order Butterworth patterns, i.e., to have the form $\rho^{-1} s_i^{\infty}$, $i = 1,2, \ldots, q$. Then the key equations (10) - (13) and (15) assume the forms

$$K_1 x_i = 0, \qquad i = 1,2,\ldots,n\text{-}q \tag{35}$$

$$(s_i I - F - GK_2)x_i = 0, \qquad i = 1,2,\ldots,n\text{-}q \tag{36}$$

$$x_i = (s_i I - F)^{-1} G\nu_i, \qquad i = 1,2,\ldots,n\text{-}q \tag{37}$$

$$\left.\begin{aligned} x_i^{\infty} &= G\,\nu_i^{\infty} \\ K_1 x_i^{\infty} &= s_i^{\infty}\,\nu_i^{\infty} \end{aligned}\right\} \qquad i = 1,2,\ldots,q \tag{38}$$

It is convenient to generalize the previous notation and define

$$X_f = \begin{bmatrix} x_1 & x_2 & \cdots & x_{n-q} \end{bmatrix}$$

$$X_\infty = \begin{bmatrix} x_1^{\infty} & x_2^{\infty} & \cdots & x_q^{\infty} \end{bmatrix}$$

$$N_f = \begin{bmatrix} \nu_1 & \nu_2 & & \nu_{n-q} \end{bmatrix}$$

$$N \quad \begin{bmatrix} \nu_1^{\infty} & \nu_2^{\infty} & & \nu_q^{\infty} \end{bmatrix}$$

$$S = \text{diag}\begin{bmatrix} s_1^{\infty} & s_2^{\infty} & \cdots & s_q^{\infty} \end{bmatrix}$$

Evaluation of K_1 , R and Q

As before, from (35) and (38) we obtain

$$K_1 = \begin{bmatrix} 0 & NS \end{bmatrix} \begin{bmatrix} X_f & X_\infty \end{bmatrix}^{-1} = NSY_\infty \tag{40}$$

$$K_1 GN = NS \tag{41}$$

(41) implies that K_1G has q non-zero eigenvalues s_i^∞ with corresponding eigenvectors ν_i^∞. From (40) it is clear that rank K_1 = q , and hence rank (K_1G) = q. Thus the remaining m-q eigenvalues of K_1G are zero. Let M denote the m×(m-q) modal matrix associated with these eigenvalues, i.e.

$$K_1GM = 0 \tag{42}$$

Then

$$K_1G = \begin{bmatrix} N & M \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} N & M \end{bmatrix}^{-1}$$

To obtain R, as before we invoke the symmetry of RKG (or KGR^{-1}). This yields

$$R^{-1} = \begin{bmatrix} N & M \end{bmatrix} \begin{bmatrix} \Gamma & 0 \\ 0 & \Theta \end{bmatrix} \begin{bmatrix} N & M \end{bmatrix}'$$

Where Γ is an arbitrary q x q diagonal matrix and Θ is an arbitrary (m-q) x (m-q) symmetric matrix, whence

$$R^{-1} = N \Gamma N' + M\Theta M' \tag{44}$$

Then from (22) we have that

$$Q = K_1' RK_1 \tag{45}$$

Although it appears from (40), (44) and (45) that the inverse problem has been solved for the case q < m, we have still to check whether the higher-order terms in the ARE are consistent with (40), (44) and (45) (as they were for the case q = m).

Consistency Check for $O(\rho)$ terms in ARE

The $O(\rho)$ terms in the ARE yield (30) which can be transformed to the form

$$K_2GR^{-1}G'K_1' + K_1GR^{-1}G'K_2' + R^{-1}G'F'K_1' + K_1FGR^{-1} = 0 \quad (46)$$

This equation clearly has a solution of the form

$$K_2GR^{-1}G'K_1' + R^{-1}G'F'K_1' = V \quad (47)$$

where V is some skew-symmetric matrix.

Plugging (40) into (47), it is evident that V must have the decomposition

$$V = LN' \quad (48)$$

where L is some mxq matrix that is yet to be determined. Moreover the skew symmetry of V implies that

$$LN' + NL' = 0 \quad (49)$$

Now the multiplication of (47) on the right by a right inverse of N' yields

$$K_2GR^{-1}G'Y_\infty'S + R^{-1}G'F'Y_\infty'S = L \quad (50)$$

Substituting of R^{-1} in the first form of (50), we find that

$$K_2\left[(GN\Gamma N'G + GM\Theta M'G')Y_\infty'S\right] = L - R^{-1}G'F'Y_\infty'S \quad (51)$$

Now using the relations $GN = X_\infty$ from (39), $Y_\infty X_\infty = I_q$ by definition, and the fact that

$$K_1GM = NSY_\infty GM = 0$$

implies that

$$Y_\infty GM = 0 \quad (52)$$

since N has full rank, equation (51) may be simplified to

$$K_2X_\infty\Gamma S = L - R^{-1}G'F'Y_\infty'S \quad (53)$$

From (38) and (53) we can then solve for K_2 as

$$K_2 = \left[N_f \quad LS^{-1}\Gamma^{-1} - R^{-1}G'F'Y_\infty\Gamma^{-1}\right]\left[X_f \quad X_\infty\right]^{-1}$$

$$= N_fY_f + (LS^{-1} - R^{-1}G'F'Y_\infty')\Gamma^{-1}Y_\infty \quad (54)$$

Since (49) always has at least the trivial solution L = 0, K_2 defined by (54) exists. However it is necessary that

$$K_2GR^{-1} = (K_2GR^{-1})' \quad (55)$$

which comprises m(m-1)/2 constraints to be satisfied by the elements of L , Γ and Θ.

Consistency Check for $O(\rho^2)$ Terms in ARE

Multiplication of the $O(\rho^2)$ equation (24) on the left by M'G' and on the right by GM yields

$$M'RK_2FGM + M'G'F'K_2'RM + M'G'K_2'RK_2GM = 0 \tag{56}$$

where we have used the equalities

$$K_1GM = 0 \quad , \quad G'P_2 = -RK_2$$

to effect a simplification.

(56) represents a further $(m-q)(m-q+1)/2$ equations to be satisfied by the elements of L, Γ and θ.

Necessary Conditions for a Solution

We first consider the case

$$mq \leq m(m+1)/2 \tag{57}$$

Then clearly, for almost all N, the only solution of (49) will be L = 0. Then we are left with the $m(m-1)/2 + (m-q)(m-q+1)/2$ equations (55), (56) in the $q + (m-q)(m-q+1)/2$ elements of Γ and θ , and a solution will exist in general only if

$$q \geq m(m-1)/2 \tag{58}$$

From (57) and (58), it follows that a solution may exist if

$$m(m-1)/2 \leqq q \leqq (m+1)/2 \tag{59}$$

We next consider the case

$$mq > m(m + 1)/2 \tag{60}$$

Now, nontrivial solutions of (49) can exist. It is clear that a condition for the existence of L, Γ and θ satisfying (49), (55) and (56) is that $mq + q + (m-q)(m-q + 1)/2 \geqq m(m + 1)/2 + m(m-1)/2$
$+(m-q)(m-q + 1)/2$

i.e.
$$q \geq m^2/(m + 1) \tag{61}$$

Since for $m \geqq 1$, (60) is satisfied whenever (61) is satisfied, we have that (61) is the alternative condition to (59) for the existence of a solution.

Now the only integral q that satisfies (61) for $m \geq 1$ is $q = m$. Also, it is easy to show that the only case with $q < m$ that is consistent with (59) is $q = 1$, $m = 2$. Thus we conclude that $q = 1$, $m = 2$ represents the only case where eigenvalue - eigenvector placement may be achieved with the same freedom as when $q = m$.

We now work out a numerical example to illustrate this case.

Example 1

Consider a plant with

$$F = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Suppose we choose the asymptotically finite eigenvalue to be $s_1 = -1$, and also choose

$$\nu_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = N_f$$

so that the corresponding eigenvector is

$$x_1 = (s_1 I - F)^{-1} G \nu_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = x_f$$

Suppose also that we choose $s_1^\infty = -1$,

$$\nu_1^\infty = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = N$$

so that the eigenvector associated with the asymptotically infinite eigenvalue is

$$x_1^\infty = G \nu_1^\infty = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = X_\infty \quad .$$

Then (40) yields

$$K_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}$$

Here $K_1G = K_1$, and hence the eigenvector M corresponding to the zero eigenvalue of K_1G is clearly

$$M = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Then from (43),

$$R^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \gamma & 0 \\ 0 & \theta \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

Since any scale factor in R can be absorbed into ρ we can take $\gamma = 1$ which gives

$$R^{-1} = \begin{bmatrix} \theta & -\theta \\ -\theta & \theta + 1 \end{bmatrix}$$

Next we find that the solution of (49) is

$$L = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence from (54) we obtain that

$$K_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \end{bmatrix} - \begin{bmatrix} \theta & -\theta \\ -\theta & \theta+1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 - \theta & -\theta \\ \theta - 2 & \theta-2 \end{bmatrix}$$

Equation (55) therefore reduces to

$$(- 1 - \theta)\ (-\theta) + (-\theta)(\theta+1) = (\theta -2)\ \theta + (\theta -2)(-\theta)$$

which is satisfied trivially.

Finally, equation (56) simplifies to

$$16\ \theta^3 - 20\theta^2 + 3\theta + 1 = 0$$

which has roots

$$\theta = 1 \ , \ (-1 \pm \sqrt{5}\,)/8.$$

The negative root can be rejected since R must be positive definite. Thus we have two solutions.

The root $\theta = 1$ corresponds to

$$R^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix},$$

i.e.

$$R = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$Q = K_1'RK_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Discussion

It is clear from the analysis of this section that, in general, solutions to the inverse problem when $q < m$ may only become possible by relaxing the requirement for eigenvector placement. That is, by making the ν_i and ν_i^{∞} free parameters instead of fixing them at the onset. Even then, there is no guarantee that a solution exists because the equations defining the solution are nonlinear unlike in the $q = m$ case. In fact, it is not even clear that arbitrary placement of just the asymptotic eigenvalues can be achieved in general.

The above conclusions are based on checking the consistency of terms only up to $O(\rho^2)$ in the ARE. Consideration of higher order terms can only yield more stringent conditions for the existence of a solution.

It should be noted, however, that if the above consistency checks are satisfied, then the control law

$$u = (\rho^{-1}K_1 + K_2)x$$

yields the desired asymptotic modal properties, although this control law may not correspond to the leading terms of an optimal control law.

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APPENDIX B

CHARACTERISTICS OF OPTIMAL LINEAR REGULATORS WITH SMALL CONTROL WEIGHTS

APPENDIX B

CHARACTERISTICS OF OPTIMAL LINEAR REGULATORS WITH SMALL CONTROL WEIGHTS

By H. R. Sirisena

ABSTRACT

A method is developed for finding the leading terms in an asymptotic expansion of the optimal control law for small weights on the controls. The corresponding expansions for the eigenvalues and eigenvectors of the optimal system are also obtained.

I. INTRODUCTION

Given the n^{th} order, m-input system

$$\dot{x} = Fx + Gu \tag{1}$$

and the performance index

$$J = \int_0^\infty (x'Qx + \rho^2 u'Ru)\, dt , \tag{2}$$

Reference 1 examines the asymptotic behavior of the optimal closed-loop eigenvalues as the scalar ρ tends to zero. References 2 and 3 consider also the asymptotic behavior of the eigenvectors, though they are primarily concerned with the inverse problem of choosing Q and R to achieve specified asymptotic eigenvalues and eigenvectors.

Here we treat the direct problem, our main objective being to determine the leading terms in the expansion of the optimal control law as a power series in ρ. The behavior of the eigenvalues and eigenvectors for small but finite values of ρ may thereby be investigated.

2. PRELIMINARY RESULTS

Without loss of generality, Reference 2, we assume that

$$Q = H'H \tag{3}$$

where H is an mxn matrix such that the system (F, G, H) is observable and minimum phase.

It is well known that the optimal control law is given by

$$K = -\rho^{-2} P^{-1} G' P \tag{4}$$

where P is the solution of the Algebraic Riccati Equation (ARE)

$$PF + F'P - \rho^{-2} PGR^{-1} G'P + H'H = 0 \tag{5}$$

The matrix P is symmetric and positive definite, and admits the power series expansion

$$P = \rho P_1 + \rho^2 P_2 + \rho^3 P_3 + \ldots \tag{6}$$

where the matrices P_i are also symmetric.

Similarly K admits the power series expansion

$$K = \rho^{-1} K_1 + K_2 + \rho K_3 + \ldots \tag{7}$$

where

$$K_i = -R^{-1} G' P_i \tag{8}$$

Plugging (6) into (5) and using (8), it is easy to show, Reference 3, that

$$K_1' R K_1 = H' H \tag{9}$$

$$P_1F + F'P_1 - K_1'RK_2 - K_2'RK_1 = 0 \tag{10}$$

$$P_2F + F'P_2 - K_1'RK_3 - K_2'RK_2 - K_3'RK_1 = 0 \tag{11}$$

3. THE CASE RANK HG = M

We distinquish between this case and the case where HG is of less than full rank because the two cases correspond to quite different asymptotic properties.

3.1 Determination of K_1

Premultiplying (9) by G' and postmultiplying by G, we obtain

$$G'K_1'RK_1G \quad = \quad G'H'HG \tag{12}$$

Since RK_1G is symmetric, (12) can be rearranged as

$$G'K_1'G'K_1'R \quad = \quad G'H'HG \tag{13}$$

whence

$$(G'K_1')^2 \quad = \quad G'H'HGR^{-1} \tag{14}$$

Now it can be shown that the product of two symmetric positive definite matrices has eigenvalues that are real and strictly positive. Thus the quantity on the RHS of (14) has real square roots, and we can write

$$G'K_1' \quad = \quad (G'H'HGR^{-1})^{\frac{1}{2}}_{-} \tag{15}$$

where the subscript '_' denotes that the square root having all its eigenvalues negative is chosen, since it is known,(Reference 3) that the eigenvalues of K_1G are all negative.

Premultiplying (9) by G', we have

$$G'K_1'RK_1 \quad = \quad G'H'H \tag{16}$$

then plugging (15) into (16) and solving for K_1, we obtain

$$K_1 \quad = \quad R^{-1}\ (G'H'HGR^{-1})^{\frac{1}{2}}_{-}\ G'H'H \tag{17}$$

3.2 Determination of K_2

On premultiplying (10) by G' and postmultiplying by G we obtain

$$G'P_1FG \ + \ G'F'P_1G \ - \ G'K_1'RK_2G \ - \ G'K_2'RK_1G \ = \ 0$$

which reduces to

$$G'K_1'(RK_2G) + (RK_2G)K_1G + RK_1FG + G'F'K_1'R \ = \ 0 \tag{18}$$

on using the relations $G'P_1 = -RK_1$ and $(RK_2G)' \ = \ RK_2G$. Since K_1G is

nonsingular, the Lyapunov equation (18) has a unique solution, say,

$$RK_2G = Z \tag{19}$$

Next we note the fact that (Reference 4)

$$P_i = -K_i'R\,(RK_iG)^{-1}\,RK_i + \bar{P}_i \tag{20}$$

is the most general P_i that satisfies (8); where $\bar{P}_i$ is any nxn symmetric matrix that satisfies

$$\bar{P}_iG = 0 \tag{21}$$

A little thought shows that the most general form of $\bar{P}_i$ is

$$\bar{P}_i = G_o'L_iG_o \tag{22}$$

where L_i is any (n-m) x (n-m) symmetric matrix, and G_o is an (n-m) x n anihilator of G.

Since K_1 is known, P_1 is partially characterized by (20) and (22), i.e., we may write

$$P_1 = \hat{P}_1 + G_o'L_1G_o \tag{23}$$

where

$$\hat{P}_1 = -K_1'R\,(RK_1G)^{-1}\,RK_1 \tag{24}$$

and L_1 is an (n-m) x (n-m) symmetric matrix that is as yet undetermined.

Now premultiplying (10) by G', and then plugging in (19) and (23), we get

$$-RK_1F + G'F'\hat{P}_1 + G'F'G_o'L_1G_o - G'K_1'RK_2 - ZK_1 = 0$$

which can be solved for K_2 to yield

$$K_2 = A_2L_1G_o + B_2 \tag{25}$$

where

$$A_2 = (RK_1G)^{-1}\,G'F'G_o' \tag{26}$$

$$B_2 = (RK_1G)^{-1}\left[G'F'\hat{P}_1 - RK_1F - ZK_1\right] \tag{27}$$

We now only need to find L_1. This may be done by plugging (23) and (25) into (10) which yields

$$(\hat{P}_1+G_o'L_1G_o)F + F'(\hat{P}_1+G_o'L_1G_o) - K_1'R(A_2L_1G_o+B_2) -$$

$$(G_o'L_1A_2'+ B_2')RK_1 = 0 \qquad (28)$$

On premultiplying (28) by $G_o^{+}{}'$ and postmultiplying by G_o^+, where G_o^+ is a right inverse of G_o, we obtain the following Lyapunov equation for L_1

$$L_1(G_oFG_o^+ - A_2'RK_1G_o^+) + (G_o^{+}{}'F'G_o' - G_o^{+}{}'K_1'RA_2)L_1 +$$

$$G_o^{+}{}'(\hat{P}_1F+F'\hat{P}_1 - K_1'RB_2-B_2'RK_1)G_o^+ = 0 \qquad (29)$$

Once L_1 is determined by solving (29), K_2 is obtained by substituting for L_1 in (25).

The next coefficient matrix K_3 can be obtained from equation (11) by methods similar to those outlined above. Similarly K_4, K_5, etc, can be obtained from the equations arising from the $O(\rho^3)$, $O(\rho^4)$, ... etc terms in the ARE (5).

3.3 Asymptotic Behavior of Eigenvalues and Eigenvectors

We now examine the asymptotic behavior of the eigenvalues and eigenvectors of the closed-loop matrix

$$F + \rho^{-1} GK_1 + GK_2 + \rho GK_3 + \ldots \qquad (30)$$

as $\rho \to 0$. Previous work (References 2 and 3) has shown that m eigenvalues tend to eigenvalues of $\rho^{-1}K_1G$ (i.e., the non-zero eigenvalues of $\rho^{-1}GK_1$), while the remaining n-m eigenvalues tend to the finite eigenvalues of $F + \rho^{-1}GK_1 + GK_2$.

The positions of the asymptotically infinite eigenvalues for small but finite values of ρ may be determined by treating $F + GK_2 + GK_3 + \ldots$ to be a perturbation of the matrix $\rho^{-1}GK_1$. Similarly, the locations of the asymptotically finite eigenvalues may be investigated by considering the perturbation $\rho GK_3 + \ldots$ of the matrix $F + \rho^{-1}GK_1 + GK_2$.

3.3.1 Asymptotically Infinite Eigenvalues

We first need to determine the matrix of eigenvectors of GK_1. Now if K_1G has the decomposition

$$K_1G = NSN^{-1} \tag{31}$$

where $S = \text{diag}(S_1^\infty, S_2^\infty \ldots S_m^\infty)$, then

$$GK_1GN = GNS$$

Hence the modal matrix associated with the nonzero eigenvalues of GK_1 is GN. From the expression (17) for K_1, it is clear that any nx (n-m) anihilator H^o of H is also an anihilator of K_1. Hence H^o qualifies as a modal matrix for the zero eigenvalues of GK_1.

Now it is easy to verify that

$$\begin{bmatrix} GN & H^o \end{bmatrix}^{-1} = \begin{bmatrix} (NS)^{-1} K_1 \\ G^o \end{bmatrix} \tag{32}$$

where G^o is the (n-m) x n anihilator of G that satisfies

$$G^oH^o = I \tag{33}$$

Hence the rows of the matrix on the RHS of (32) are the dual eigenvectors of GK_1.

Finally, we apply the following standard formulas (Reference 4) for perturbations of eigenvalues and eigenvectors of a nxn matrix A due to a small perturbation δA

$$\delta\lambda_i = v_i \delta A u_i \tag{34}$$

$$\delta\lambda_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \left[(v_j \delta A u_i)/(\lambda_i - \lambda_j) \right] u_j \tag{35}$$

Where λ_j, $j = 1,2,\ldots,n$ are the eigenvalues of A and u_j, v_j, $j = 1,2,\ldots,n$ are an orthonormal set of eigenvectors and dual eigenvectors of A. We have that the perturbation δs_i^∞ of the eigenvalue s_i^∞ of GK_1 is given by

$$\delta s_i^\infty = \rho(NS)_i^{-1} K_1 (F+GK_2) G\nu_i \tag{36}$$

where $(NS)_i^{-1}$ denotes the i-th row of $(NS)^{-1}$ and v_i denotes the i-th column of N. Hence the two leading terms in the power series expansion of the asymptotically infinite eigenvalues are

$$\rho^{-1} s_i^{\infty} + (NS)_i^{-1} K_1 (F + GK_2) G v_i \qquad (37)$$

We also have that the perturbations in the corresponding eigenvectors x_i^{∞} are

$$\delta x_i^{\infty} = \rho \left\{ \sum_{\substack{j=1 \\ j \neq i}}^{m} \left[(N\ S)_j^{-1} K_1 (F + GK_2) x_i^{\infty} / (s_i^{\infty} - s_j^{\infty}) \right] x_j^{\infty} + \sum_{j=1}^{n-m} (G^0 F x_i^{\infty} / s_i^{\infty}) H_j^0 \right\} \qquad (38)$$

where H_j^o denotes the j-th column of H^0.

3.3.2 Asymptotically Finite Eigenvalues

In Reference 3 it is shown that the asymptotically finite eigenvalues are the n-m eigenvalues s_i, i = 1,2,...,n-m of the matrix $F + \rho^{-1} GK_1 + GK_2$ that are associated with eigenvectors x_i that lie in the kernel of K_1, i.e. are such that

$$K_1 x_i = 0, \quad i = 1,2,\ldots,n\text{-}m \qquad (39)$$

The remaining m eigenvalues are of course asymptotically associated with the eigenvectors x_i^{∞}, i = 1,2,...,m.

The dual eigenvectors y_i, i = 1,2,...,n-m associated with the asymptotically finite eigenvalues s_i are thus given by

$$y_i = \left[x_1\ x_2 \ldots x_{n-m}\ x_1^{\infty}\ x_2^{\infty} \ldots x_m^{\infty} \right]_i^{-1} \qquad (40)$$

where the subscript i denotes "i-th row of".

Then applying the formula (34) we find that the perturbation in s_i is given by

$$\delta s_i = \rho\, y_i G K_3 x_i, \quad i = 1,2,\ldots,n\text{-}m \qquad (41)$$

Similarly, the perturbations in the corresponding eigenvectors may be found by applying equation (35).

4. THE CASE RANK HG<M

We consider only the case: rank H = rank HG<M.

4.1 Determination of K_1

The analysis of Section 3.1 holds up to equation (16) but the matrix inverse in the expression (17) for evaluating K_1 does not exist. However, this difficulty can be circumvented as follows.

Let $G'K_1'$, given by (15), have the decomposition

$$G'K_1' = LM \tag{42}$$

where L,M are mxq and qxm matrices respectively, where q = rank $G'K_1'$. It is then clear that K_1 can be decomposed as

$$K_1 = M'T \tag{43}$$

since G has full rank.

Now plugging (42) and (43) into (16), we have that

$$LMRM'T = G'H'H$$

whence we obtain

$$T = (MRM')^{-1}L^{+}G'H'H \tag{44}$$

where L^{+} is a left inverse of L. Hence

$$K_1 = M'(MRM')^{-1}L^{+}G'H'H \tag{45}$$

4.2 Determination of K_2

Finding K_2 is much more complicated than when rank HG = m, and no general procedure has been devised. Essentially, one has to solve the simultaneous nonlinear equations (8), (10) and (11) while invoking the symmetry of the P_i. The steps involved are best illustrated by means of an example.

Example 1

Consider

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad G = I, \quad H = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad R = I .$$

First, by evaluating (39), we obtain that

$$K_1 = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Then from (8) we find that $P_1 = -K_1$. Also, K_2 is symmetric since P_2 is symmetric. Therefore let

$$K_2 = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$$

Then equation (10) can be written

$$\begin{bmatrix} 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} - \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} x & y \\ y & z \end{bmatrix} -$$

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = 0$$

This yields the equations

$$x + y = 0 \qquad \text{(i)}$$

$$x + 2y + z = 1 \qquad \text{(ii)}$$

$$y + z = 1 \qquad \text{(iii)}$$

of which only two are linearly independent.

Next we consider equation (11) which we multiply on the left by $\begin{bmatrix} 1 & -1 \end{bmatrix}$, an anihilator of K_1', and on the right by $\begin{bmatrix} 1 & -1 \end{bmatrix}'$ to obtain

$$8x^2 + 8x + 1 = 0 \qquad \text{(iv)}$$

where we have used the fact that $P_2 = -K_2$.

Solving (iv), we get

$$x = (-2 \pm \sqrt{2})/4 \qquad \text{(v)}$$

whence from (i) and (iii)

$$y = (2 \mp \sqrt{2})/4 \qquad \text{(vi)}$$

$$z = (2 \pm \sqrt{2})/4 \qquad \text{(vii)}$$

To select the correct sign, we note that the asymptotically finite eigenvalue, which is the eigenvalue of $F + GK_2$ whose associated eigenvector lies in the kernel of K_1 is

$$s_1 = -1 - 2x = \mp 1/\sqrt{2}$$

Thus on the basis of stability, we choose the + signs in (v) and (vii) and hence the - sign in (vi), so that we have that

$$K_2 = \begin{bmatrix} (-2 + \sqrt{2})/4 & (2 - \sqrt{2})/4 \\ (2 - \sqrt{2})/4 & (2 + \sqrt{2})/4 \end{bmatrix} .$$

4.3 Asymptotic Behavior of Eigenvalues and Eigenvectors

Once K_1 and K_2 have been determined, the asymptotic behavior of eigenvalues and eigenvectors may be analyzed in much the same way as in Section 3.3. The main difference is that equation (32) no longer applies. Hence (36) becomes

$$\delta s_i^{\infty} = \rho \left[GN \;\; H^0\right]_i^{-1} (F + GK_2)\, Gv_i \qquad (46)$$

where N is now the modal matrix associated with only the non-zero eigenvalues of K_1G. A similar modification has to be made in equation (38).

In the case of the asymptotically finite eigenvalues, equation (41) continues to apply except that equation (40) defining y_i must be modified to reflect the fact that there are now fewer than m infinite eigenvalues.

5. CONCLUSION

Given a system and an associated quadratic performance index, a method has been developed for finding the leading terms in an asymptotic expansion of the optimal control law. The asymptotic behavior of the eigenvalues and eigenvectors of the optimal system are thereby partially characterized. The results are less than general in that the case rank $H = m$, rank $HG < m$ has not been treated.

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APPENDIX C

GUARANTEED MARGINS FOR LQG REGULATORS

APPENDIX C

GUARANTEED MARGINS FOR LQG REGULATORS

By John C. Doyle

INTRODUCTION

Considerable attention has been given lately to the issue of robustness of linear-quadratic (LQ) regulators. The recent work by Safonov and Athans [1] has extended to the multivariable case the now well-known guarantee of 60° phase and 6 db gain margin for such controllers. However, for even the single-input, single-output case there has remained the question of whether there exist any guaranteed margins for the full LQG (Kalman filter in the loop) regulator. By counter-example, this memo answers that question; there are none.

A standard two-state single-input single-output LQG control problem is posed for which the resulting closed-loop regulator has arbitrarily small gain margin.

EXAMPLE

Consider the following:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + v$$

where (x_1, x_2), u, and y denote the usual states, control input and measured output, and where w and v are Gaussian white noises with intensities $\sigma > 0$ and 1, respectively.

Let performance integral have weights

$$Q = qCC^T = q \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} , \quad q > 0$$

and $R = 1$

Note that the estimation and control problems have identical (dual) solution matrices.

It can be shown analytically that the optimal gain vector g in u = -g'x may be written as a function of q as

$$g = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(2 + \sqrt{4 + q}\right)$$

A similar relation holds between the optimal filter gain k and σ. For simplicity, let

$$g = f \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and $$k = d \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where $f = 2 + \sqrt{4 + q}$ and $d = 2 + \sqrt{4 + \sigma}$

Suppose that the resulting closed-loop controller has a scalar gain m (nominally unity) associated with the input matrix. Only the nominal value of this gain is known to the filter. The full system matrix then becomes

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -mf & -mf \\ d & 0 & 1-d & 1 \\ d & 0 & -d-f & 1-f \end{bmatrix}$$

Evaluation of the characteristic polynomial is rather tedious but reveals that only the last two terms are functions of m. The linear term is

$$d + f \; -4 + 2(m-1)df$$

and the constant term is

$$1 + (1-m)df.$$

A necessary condition for stability is that both terms be positive. It is easy to see that for sufficiently large d and f (or q and σ),

the system is unstable for arbitrarily small perturbations in m in either direction. Thus, by choice of q and σ the gain margins may be made arbitrarily small.

It is interesting to note that the margins deteriorate as control weight gets small and/or system driving noise gets large. In modern control folklore, these have often been considered ad hoc means of improving sensitivity.

It is also important to recognize that vanishing margins are not only associated with open loop unstable systems. It is easy to construct minimum phase, open loop stable counterexamples for which the margins are arbitrarily small.

The point of these examples is that LQG solutions, unlike LQ solutions, provide no global system - independent guaranteed robustness properties. Like their more classical colleagues, modern LQG designers are obliged to test their margins for each specific design.

It may, however, be possible to improve the robustness of a given design by relaxing the optimality of the filter with respect to error properties. A promising approach appears to be the introduction of certain fictitious system noises in the filter design procedure. This approach will be the topic of future papers.

REFERENCES

1. M. G. Safonov and M. Athans, "Gain and Phase Margins for Multiloop LQG Regulators", IEEE Trans. on Auto. Cont., April 1977.

APPENDIX D

CHARACTERIZATION OF UNCERTAINTY

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CHARACTERIZATION OF UNCERTAINTY

An important aspect of the total system design procedure is the characterization of the uncertainty associated with using a model for a physical plant. A design model must extract the important features of the plant but it is neither possible nor desirable to represent the plant in its full complexity.

However, when the model is being used for feedback design, it is important to have some knowledge of the possible differences between the actual plant upon which the controller must operate and the model used in the design stage. These differences arise from parameter variations, unmodelled dynamics, and approximations due to lumping and linearization. (This paper does not treat uncertainty due to disturbances or noise.) The nature of the uncertainty will dictate how feedback will be used to reduce that uncertainty.

In practice, the engineer must rely heavily on physical intuition and experience to determine the nature of uncertainty, as well as the recommendations of other engineers perhaps more intimately familiar with the process to be controlled. However, certain features of physical systems seem to be fairly universal leading to certain common characteristics of uncertainty when the model is linear:

1. The uncertainty grows with increasing frequency due to unmodelled high frequency dynamics, lumping and linearization of nonlinear components. This is embodied in the familiar notion of a "model being good out to frequency ω_0."

2. A particular aspect of this growth in uncertainty is that phase becomes totally unknown beyond some frequency. This is a critical limitation on the bandwidth of the feedback design.

3. Actuators tend to have significant nonlinearities due to rate and magnitude limits and dead bands. The linear models used for such actuators will be entirely inadequate when the actuators are driven too hard. This places further restrictions on loop bandwidth.

4. Some subsystems may be very accurately modelled by linear systems with parameter uncertainty. The parameter variations may be unknown but may be bounded or have some probability distribution associated with them. If these variations are regular and identifiable, some form of an adaptive scheme may be called for. In any case, a controller must be insensitive or robust with respect to such variations.

5. Quite often there is uncertainty which cannot be attributed directly to any particular parameter variation or specific modelling assumption. This can occur when frequency response data is used to derive models or when the exact physics of a process is poorly understood (for whatever reason). Although only a crude model may be available, it is often possible to accurately bound the total uncertainty of the input/output properties of such systems by envelopes of transfer functions.

Ideally, a scheme for the characterization of uncertainty should provide the engineer with:

1. A capability for dealing with information in the forms indicated above with all its vagueness.

2. A capability to exploit and blend the computational power of the computer with the pattern recognition and intuitive capability of the human.

3. A capability to use precise data on parameter variations when available.

4. A set of tools for characterization of uncertainty which form a natural part of the total set used for the entire modelling and design effort.

5. A means of representing the system which extracts the essential features necessary for feedback design.

6. Tools which provide insight into system behavior.

7. Tools which are reliable in that they neither neglect any important sources of uncertainty nor lead to overly conservative designs.

For single input/single output systems, the usual techniques of classical and modern control have proven fairly adequate. Of particular value are the frequency domain techniques involving Bode, Nyquist, and Nichols charts.

The generalization of these techniques appropriate to multiloop feedback systems appears to involve the use of singular valves.

APPENDIX E

ROBUSTNESS OF MULTILOOP LINEAR FEEDBACK SYSTEMS

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ROBUSTNESS OF MULTILOOP LINEAR FEEDBACK SYSTEMS

BY J.C. Doyle

I. INTRODUCTION

A critical property of feedback systems is their robustness; that is, their ability to maintain performance in the face of uncertainties. In particular, it is important that a closed-loop system remain stable despite differences between the model used for design and the actual plant. These differences result from variations in modelled parameters as well as plant elements which are either approximated, aggregated, or ignored in the design model. The robustness requirements of a linear feedback design are often specified in terms of desired gain and phase margins and bandwidth limitations associated with loops broken at the input to the plant actuators ([1], [2]). These specifications reflect in part the classical notion of designing controllers which are adequate for a set of plants constituting a frequency-domain envelope of transfer functions [3]. The bandwidth limitation provides insurance against the uncertainty which grows with frequency due to unmodelled or aggregated high frequency dynamics.

The Nyquist or Inverse Nyquist diagram (polar plots of the loop transfer function) provides a means of assessing stability and robustness at a glance. For multiloop systems, scalar Nyquist diagrams may be constructed for each loop individually providing some measure of robustness. Unfortunately, the method may ignore variations which simultaneously affect multiple loops.

There are a number of other possible ways to extend the classical frequency-domain techniques. One involves using compensation or feedback to decouple (or approximately decouple) a multiloop system into a set of scalar systems which may be treated with scalar techniques (i.e., "Diagonal Dominance", Rosenbock [4]). Another method uses the eigenvalues of the loop transfer matrix (G(s) in Figure 1) as a function of frequency (i.e., "Characteristic Loci", MacFarlane, et. al. [5] , [6]). While these methods provide legitimate tools for dealing with multivariable systems, they can lead to highly optimistic conclusions about the robustness of multiloop feedback designs. Examples in Section III will demonstrate this.

This paper develops an alternative view of multiloop feedback systems which exploits the concepts of singular values, singular vectors, and the spectral norm of a matrix. ([7] - [10]). This approach leads to a reliable method for analyzing the robustness of multivariable systems.

Section II presents a basic theorem on robustness and sensitivity properties of linear multiloop feedback systems. Multivariable generalizations of the scalar Nyquist, Inverse Nyquist and Bode analysis methods are then developed from this same result.

Two simple examples are analyzed in Section III using the tools of Section II. As promised, the inadequacies of the existing approaches outlined earlier will be made clear.

Section IV contains a discussion of some of the implications of this work.

The goal of this paper is to focus on the analysis of robustness and sensitivity aspects of linear multiloop feedback systems. Some new approaches emerge which yield important insignts into their behavior. The mathematical aspects of these topics are fairly mundane at best, so rigor and generality are almost always sacrificed for simplicity.

Preliminaries and Definitions

A brief discussion of singular values and vectors follows. Although the concepts apply more generally, only square matrices will be considered in this paper. A more thorough discussion of these topics may be found in [7] - [10] .

The singular values σ_i of a complex n x n matrix A are the non-negative square roots of the eigenvalues of A^*A where A^* is the conjugate transpose of A. Since A^*A is Hermitian, its eigenvalues are real. The (right) eigenvectors v_i of A^*A and r_i of AA^* are the right and left singular vectors, respectively, of A. These may be chosen such that

$$\sigma_i r_i = A v_i \quad , \qquad i = 1, \ldots n$$
$$\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_n \tag{1}$$

and the $\{r_i\}$ and $\{v_i\}$ form orthonormal sets of vectors.

It is well known that

$$A = R \, \Sigma \, V^* \tag{2}$$

where R and V consist of the left and right singular vectors, respectively, and $\Sigma = \text{diag.}\ (\sigma_1, \ldots, \sigma_n)$. The decomposition in (2) is called the singular value decomposition.

Denote

$$\underline{\sigma}\ (A) = \min_{||x||=1} ||Ax|| = \sigma_1 \tag{3}$$

and

$$\overline{\sigma}\ (A) = \max_{||x||=1} ||Ax|| = ||A||_2 = \sigma_n \tag{4}$$

where $||x|| = (x^* x)^{\frac{1}{2}}$ and $||\cdot||_2$ is the spectral norm.

The singular values are important in that they characterize the effect that A has as a mapping on the magnitude of the vectors x. The singular values also give a measure of how "close" A is to being singular (in a parametric sense). In fact, the quantity

$$\frac{\underline{\sigma}}{\overline{\sigma}}$$

is known as the condition number with respect to inversion [9]. The eigenvalues of A do not in general give such information. If λ is an eigenvalue of A, then

$$\underline{\sigma} \leq |\lambda| \leq \overline{\sigma}$$

and it is possible for the smallest eigenvalue to be much larger than $\underline{\sigma}$.

II. BASIC RESULTS

Consider identity the feedback system in Fig. 2 where G(s) is the rational loop transfer matrix and L(s) is a perturbation matrix, nominally zero, which represents the deviation of G(s) from the true plant. While this deviation is unknown, there is usually some knowledge as to its size. A reasonable measure of robustness for a feedback system is the magnitude of the otherwise arbitrary perturbation which may be tolerated without instability. The following theorem characterizes robustness in this way. The "magnitude" of L(s) is taken to be the spectral norm. Only stable perturbations are considered since no feedback design may be made robust with respect to arbitrary unmodeled unstable poles.

Robustness theorem: Consider the perturbed system in Fig. 2 with the following assumptions

i) G(s) and L(s) are nxn rational square matrices,

ii) $\det(G(s)) \not\equiv 0$

iii) L(s) is stable

iv) the nominal closed loop system
$H = G(I+G)^{-1}$
is stable.

Under these assumptions the perturbed system is stable if

$$\underline{\sigma}\,(I + G(s)^{-1}) > \bar{\sigma}\,(L(s)) \tag{5}$$

for all s in the classical Nyquist D-contour (defined below)

Proof:

It is well known [4] that since G is invertible

$$\det(H(s)^{-1}) = \det(I + G(s)^{-1}) = \frac{\psi_1(s)}{\phi_1(s)} \tag{6}$$

where $\psi_1(s)$ is the nominal closed-loop characteristic polynomial and $\phi_1(s)$ is the transmission zero polynomial of G [11] .

For the perturbed system

$$\det(I + G(s)^{-1} + L(s)) = \frac{\psi_2(s)}{\phi_1(s)\psi_3(s)} \tag{7}$$

where $\psi_2(s)$ is the perturbed closed-loop characteristic polynomial and $\psi_3(s)$ is the characteristic polynomial of L(s).

Let D be a large contour in the s-plane consisting of the imaginary axis from -jR to +jR, together with a semicircle of radius R in the right half-plane. The radius R is chosen large enough so that all finite roots of $\psi_2(s)$ have magnitude less than R.

Let the contour Γ_0 be the image of D under the map $\psi_1(s)\det(I + G(s)^{-1})$. Since H is stable, it follows from the principle of the argument that Γ_0 will not encircle the origin.

Define the map

$$\gamma(q,s) = \phi_1(s)\det(I + G(s)^{-1} + qL(s)), \quad q \text{ real} \tag{8}$$

and let $\gamma(q,s)$ map D into the Contour $\Gamma(q)$ for fixed q, $0 \leq q \leq 1$. The map $\gamma(q,s)$ may be written as

$$\begin{aligned}\gamma(q,s) &= \frac{\psi_1(s)\psi_3(s) + q\theta_1(s) + \dots + q^n\theta_n(s)}{\psi_3(s)} \\ &= \frac{\psi_4(q,s)}{\psi_3(s)} \quad .\end{aligned} \tag{9}$$

Clearly, since $\Gamma(o) = \Gamma_0$, it does not encircle the origin. Since the roots of ψ_4 are algebraic functions of q, they are continuous in q [12]. Thus the only way that the perturbed contour $\Gamma(1)$ can encircle the origin is for

$$\det\,(I + G(s)^{-1} + qL(s)) = 0 \tag{10}$$

for some s in D and some q on the interval $o \leq q \leq 1$. (Recall that $\psi_3(s)$ has no right half-plane roots). When (10) is satisfied then $\underline{\sigma}\,(I + G^{-1} + qL)$ must also be zero. However, as a consequence of (5)

$$\begin{aligned} \underline{\sigma}\,(I + G^{-1} + qL) &\geq \underline{\sigma}\,(I + G^{-1}) - q\,\bar{\sigma}(L) \\ &\geq \underline{\sigma}\,(I + G^{-1}) - \bar{\sigma}(L) \\ &> 0 \quad . \end{aligned} \tag{11}$$

Thus $\Gamma(q)$ does not encircle the origin for $o \leq q \leq 1$. In particular, the perturbed contour $\Gamma(1)$ does not encircle the origin, and the perturbed closed-loop system is stable. $\square$

Similar theorems hold for additive rather than multiplicative perturbations (with I + G substituted for $I + G^{-1}$) as well as a number of other configurations.

This theorem points out the importance of singular values. In particular, the smallest singular value $\underline{\sigma}(I + G(j\omega)^{-1})$ gives a reliable frequency-dependent measure of robustness. Stability is guaranteed for all perturbations whose spectral norm is less than $\underline{\sigma}$. As will be seen in the examples, eigenvalues do not give a similarly reliable measure.

The singular values also have useful graphical interpretations. Consider the dyadic expansion

$$\begin{aligned} H^{-1} = I + G^{-1} &= \sum_{i=1}^{n} \sigma_i r_i v_i^* \\ \sigma_1 \leq \sigma_2 \leq &\cdots \leq \sigma_n \end{aligned} \tag{12}$$

where the σ_i, r_i and v_i are the singular values, and left and right singular vectors, respectively of $I + G^{-1}$. This is an alternative form of the singular value decomposition in equation (2).

It has been shown [5] that the eigenvalues and eigenvectors of a rational matrix are continuous (through generally not rational) functions of frequency. Since singular values and vectors are just special cases, $\sigma_i(j\omega)$, $r_i(j\omega)$ and $v_i(j\omega)$ are also continuous functions of ω.

Since

$$H = (I + G^{-1})^{-1} = \Sigma \frac{1}{\sigma_i} v_i r_i^* \tag{13}$$

the values $1/\sigma_1(j\omega)$ and $1/\sigma_n(j\omega)$ give the maximum and minimum possible magnitude responses to an input sinusoid at frequency ω. Eigenvalues give no such information. In this sense, a plot of these singular values vs. frequency may be thought of as a multivariable generalization of the Bode gain plot. Plots of this type will be referred to as σ-plots.

Another useful graphical interpretation analogous to the scalar Inverse Nyquist diagram may be constructed by noting that

$$\begin{aligned} G^{-1} &= \Sigma\sigma_i r_i v_i^* - I \\ &= \Sigma\sigma_i r_i v_i^* - \Sigma v_i v_i^* \\ &= \Sigma(\sigma_i r_i - v_i) v_i^* \\ &= \Sigma\beta_i g_i v_i^* \end{aligned} \tag{14}$$

where $\beta_i g_i = \sigma_i r_i - v_i$ with β_i real and $||g_i|| = 1$ for all i. (The g_i's do not necessarily form an orthonormal set.)

The quantities in (14) at some frequency ω_0 are related as in diagram in Fig. 3a. Since v_i is of unit length a complex plane may be constructed as

in Fig. 3b, to lie in the plane formed by the triangle of v_i, $\sigma_i r_i$ and $\beta_i g_i$.

Define z_i to be the complex number at the point of the triangle as in Fig. 3c. Then, by rotating the complex plane with the triangle as a function of frequency, a $z_i(j\omega)$ may be obtained which is a continuous function of ω (Fig. 3d). This allows the important quantities in (13) and (14), that is, the σ_i and β_i to be represented in convenient graphical form. As noted in Fig. 3d, there is an ambiguity to z_i depending on which side the plane is viewed. (To be more precise, the z_i represent a multivalued function of s which could be defined on appropriate Riemann sheets. However, this will be ignored.) The z_i may be calculated by finding the roots of the quadratic equation

$$z_i^2 + (1 + \beta_i^2 - \sigma_i^2)z_i + \beta_i^2 = 0 \tag{15}$$

By plotting the $z_i(j\omega)$ c$_i$ = 1,...m) for frequencies of interest a plot analogous to the scalar Inverse Nyquist plot is generated. While phase does not have the conventional meaning on these plots, the more important notion of distance from the critical point preserves its importance. These plots will be referred to as z-plots.

Concepts such as M-circles are also obvious in this context. The minimum value of M is given by

$$M_m = \max_{\omega} (1/\sigma_1(j\omega))$$

Similar results may be obtained for additive perturbations by working with $I + G$ rather than $I + G^{-1}$. In this case a diagram is generated which is analogous to the scalar Nyquist diagram. A number of other configurations may be handled as well.

Note that singular values offer no encirclement condition to test for right half-plane poles. Another test must be made for absolute stability but this presents no obstacle as many simple techniques exist for doing this. Once stability is determined the various approaches presented in this Section may be used to reliably analyze robustness.

III. EXAMPLES

Two simple examples are presented and analyzed using the approaches developed in the previous section. For the purpose of comparison, the methods of loop-breaking, direct eigenvalue analysis of G, and diagonalization by compensation are also used. The advantage of the interpretation of robustness given in this paper is clearly illustrated.

The first example is an oscillator with open loop poles at $\pm 10j$ and both closed loop poles at -1. There are no transmission zeros. The loop transfer function is

$$G(s) = \frac{1}{s^2+100} \begin{bmatrix} s-100 & 10(s+1) \\ \\ -10(s+1) & s-100 \end{bmatrix} \qquad (16)$$

By closing either loop (the system is symmetric) as in Figure 4, the transfer function for the other loop is

$$g(s) = \frac{1}{s}$$

which indicates ∞ db gain margin in both directions and 90° phase margin in each loop (with the other closed). This is very misleading, however.

The z-plot for this example is shown in Figure 5. It may appear somewhat peculiar, since it is not a plot of a rational function.

The important feature is the proximity of the plot to the critical point, indicating a lack of robustness.

The apparent discrepency between these two robustness indications can be easily understood by considering a diagonal perturbation

$$L = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \tag{17}$$

where k_1 and k_2 are constants.

Then regions of stability and instability may be plotted in the (k_1, k_2) plane as has been done in Figure 6. The open loop point corresponds to $k_1 = k_2 = -1$ and nominal closed loop point corresponds to $k_1 = k_2 = 0$. Breaking each loop individually examines stability along the k_1, k_2 axes where robustness is good, but misses the close unstable regions caused by simultaneous changes in k_1 and k_2. Thus, single loop analysis is not a reliable way of testing robustness.

The second example is a two dimensional feedback system with open-loop poles at -1 and -2 and no transmission zeroes.

The loop transfer matrix is

$$G(s) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} -47s + 2 & 56s \\ -42s & 50s + 2 \end{bmatrix} \tag{18}$$

Assume that identity feedback is used, with closed-loop poles at -2 and -4. This system may be diagonalized by introducing constant compensation. Let

$$U = \begin{bmatrix} 7 & 8 \\ 6 & 7 \end{bmatrix} \tag{19}$$

and

$$V = U^{-1} = \begin{bmatrix} 7 & -8 \\ -6 & 7 \end{bmatrix} . \tag{20}$$

Then letting

$$\hat{G} = VGU = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{2}{s+2} \end{bmatrix} , \tag{21}$$

the system may be rearranged so that

$$\begin{aligned} H &= G(I + G)^{-1} \\ &= U\hat{G}V(I + U\hat{G}V)^{-1} \\ &= U\hat{G}(I + \hat{G})^{-1}V \\ &= U\left[\hat{G}(I + \hat{G})^{-1}\right] V. \end{aligned} \tag{22}$$

This yields a diagonal system that may be analyzed by scalar methods. In particular under the assumption of identity feedback

$\hat{G}$ represents the new loop transfer matrix. Because U and V represent a similarity transformation, the diagonal elements of $\hat{G}$ are also the eigenvalues of G so that the decoupling or dominance approach and eigenvalue or characteristic loci approach would generate the same Nyquist or Inverse Nyquist plot shown in Figure 7. Only a single locus is shown since the contours of $1/(s + 1)$ and $2/(s + 2)$ are identical. The tempting conclusion that might be reached from these plots is that the feedback system is emminently robust with apparent margins of $\pm \infty$ db in gain and 90^{o}+ in phase. The closed-loop pole locations would seem to support this.

This conclusion, however, would be wrong. The z-plot for $I + G^{-1}$ is shown in Figure 8 and there is clearly a serious lack of robustness. The (k_1, k_2) - plane stability plot for this example is shown in Figure 9. Neither the diagonal dominance nor eigenvalue approaches indicate the close proximity of an unstable region. This failure can be attributed to two causes.

First, the eigenvalues of a matrix do not, in general, give a reliable measure of its distance (in a parametric sense) from singularity, and so computing the eigenvalues of G(s) (or I + G(s)) does not give an indication of robustness. Using eigenvalues rather than singular values will always detect unstable regions that lie along the $k_1 = k_2$ diagonal, but may miss regions such as the one in Figure 9.

Second, when compensation and/or feedback is used to achieve dominance, the "new plant" includes this compensation and feedback. Because of this, no reliable conclusions may be drawn from this "new plant" concerning the robustness of the final design with respect to variations in the actual plant. It is important to

evaluate robustness where there is uncertainty.

Another important property of multiloop feedback is that, unlike scalar feedback, pole locations alone are not reliable indicators of robustness. This was demonstrated in the last example and may be explained as follows. Consider a state feedback problem where the plant is controllable from each of two inputs. One input may be used to place the poles far into the right half plane and the other used to bring them back to the desired location. Such a high-gain control design of "opposing" loops will be extremely sensitive to parameter variations regardless of the nominal pole locations.

It is interesting to examine the σ-plot of $H = G(I + G)^{-1}$ for the second example shown in Figure 10. Recall that the singular values of H are equal to the inverses of the singular values of $I + G^{-1}$. There is a rather large peak in the frequency response at approximately 3 radians. This could not occur in scalar unity feedback systems without there being a pole relatively near the imaginary axis. It can happen in multiloop systems because of the high gains possible without correspondingly large pole movement.

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IV. FURTHER COMMENTS AND CONCLUSIONS

The approach to the analysis of robustness presented here appears to yield useful insight into the properties of multiloop feedback systems which may provide the basis for a multivariable stability specification analogous to gain and phase margins for scalar systems. One possible difficulty with the approach is that it can lead to overly pessimistic views of robustness because it considers perturbations which may not be physically possible. This problem exists as well with gain and phase margin evaluations. Of course, some of this difficulty can be alleviated by examining the specific perturbations leading to instability. These may be easily computed from equation (12). On the other hand, it might be argued that some healthy pessimism would be refreshing in the field of multivariable linear control research.

Although for simplicity's sake only rational transfer functions were considered the results in this paper should extend to nonrational transfer functions. In practical application it should be possible to use frequency response data directly.

The results may also be extended to include nonlinear perturbations by exploiting the general stability theory developed by Safonov [13] . In this setting, nonlinearities may be loosely viewed as linear time-invariant elements with time-varying parameters. A mathematically more rigorous treatment of these issues may be found in Zames([14] , [15]), as well as in [13] .

The results in this paper concerning dominance methods and use of characteristic loci of the loop transfer matrix are not meant to imply that design procedures employing these methods are useless. However, simply designing "in the frequency domain" is no guarantee that resulting controllers will have no undesirable properties.

Multivariable diagrams such as the σ and z- plots appear to be amenable to implementation on a computer with graphic and plotting capability. Singular

values and vectors are particularly easy quantities to compute [16]. This should facilitate their active use in multiloop feedback design procedures. The question naturally arises concerning the implications of the singular value approach for robust synthesis. Certainly, this appears to be a promising area for research.

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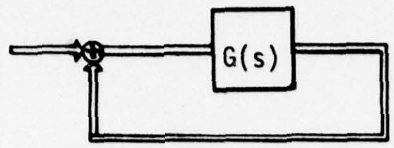

Fig. 1 Multiloop Feedback System

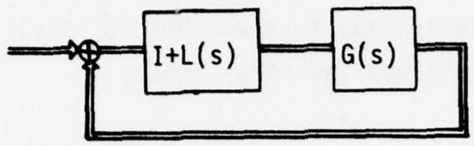

Fig. 2 Perturbed System

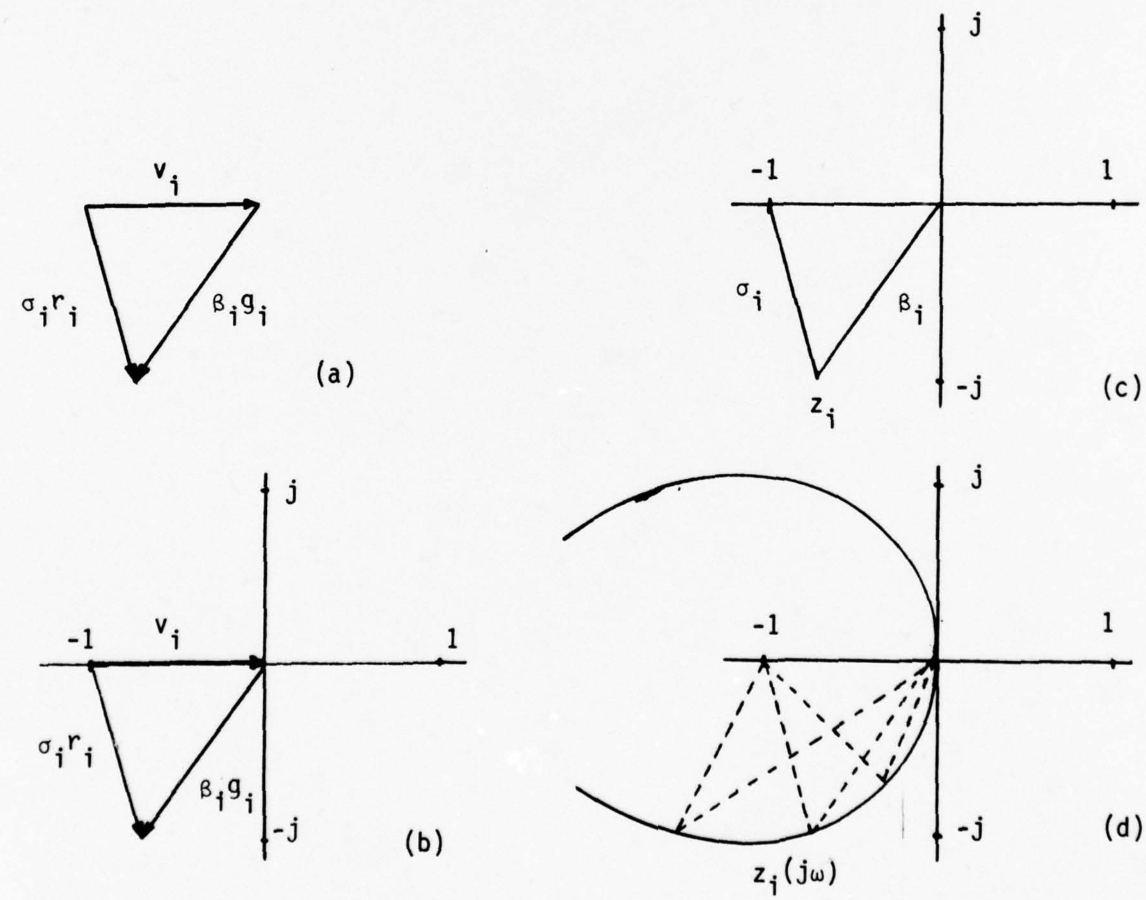

Fig. 3 Construction of z-plot

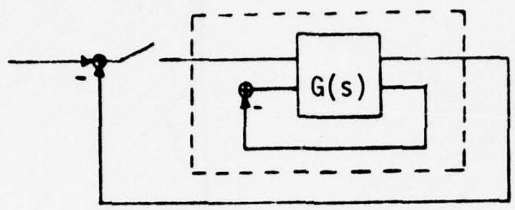

Fig. 4 Analysis by Loop-Breaking

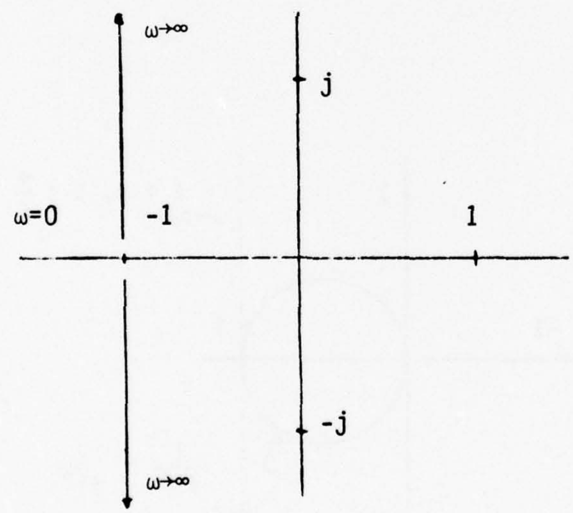

Fig. 5 Example 1 z-plot

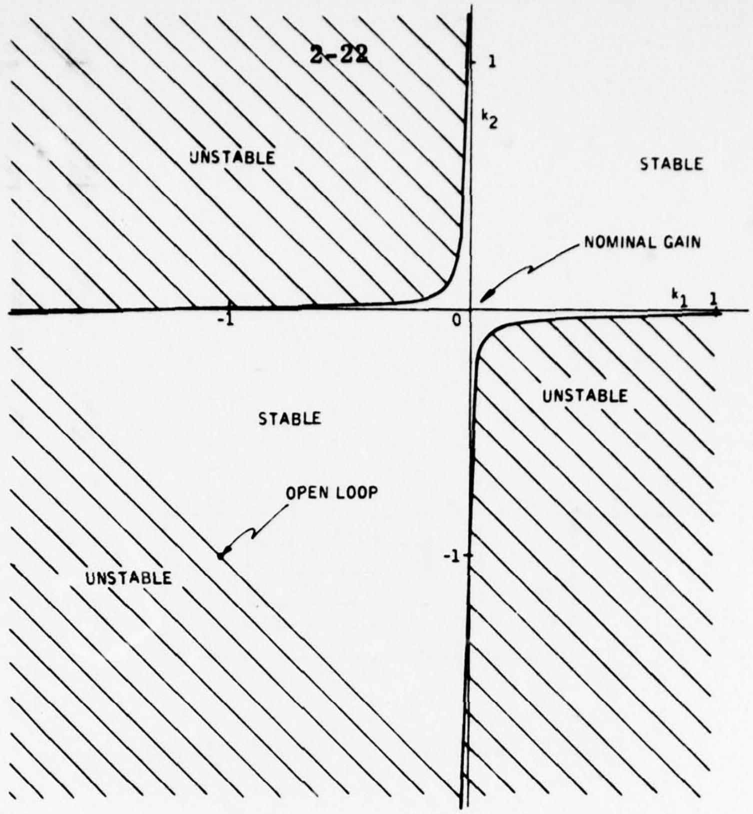

Fig. 6 Example 1 Stability Domain

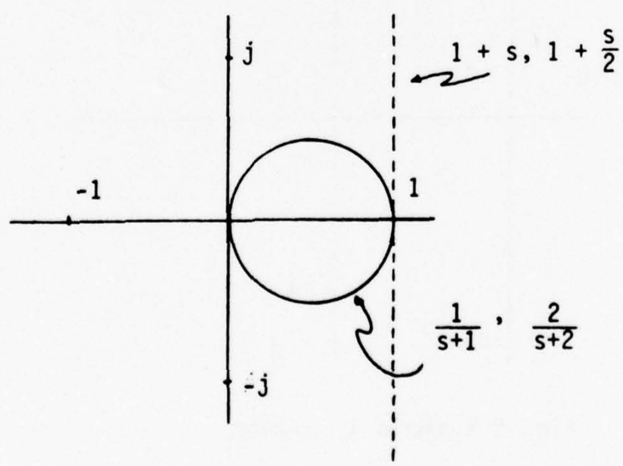

Fig. 7 Example 2 Nyquist and Inverse Nyquist Diagram

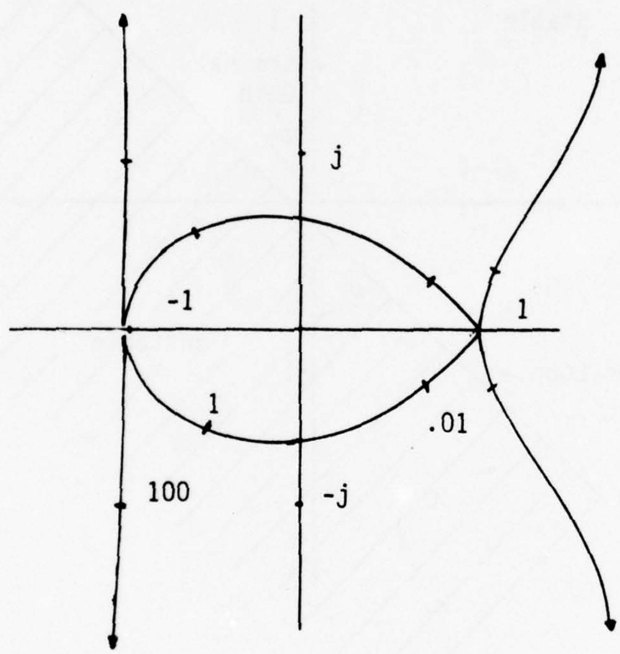

Fig. 8 Example 2 z-plot

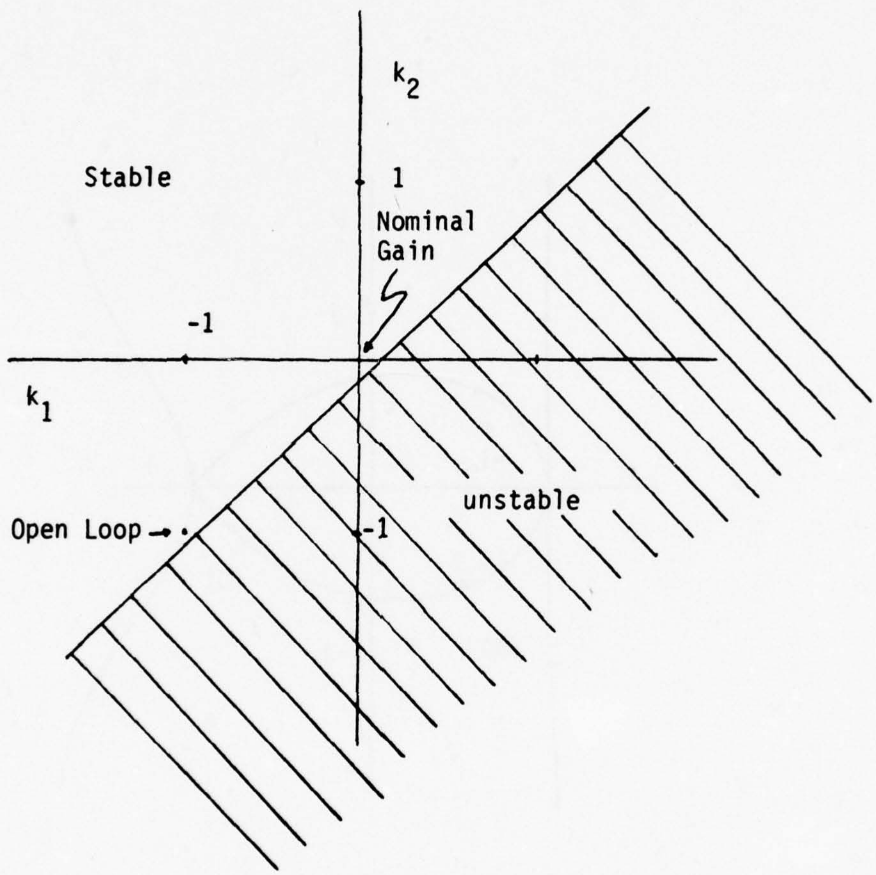

Fig. 9 Example 2 Stability Domain

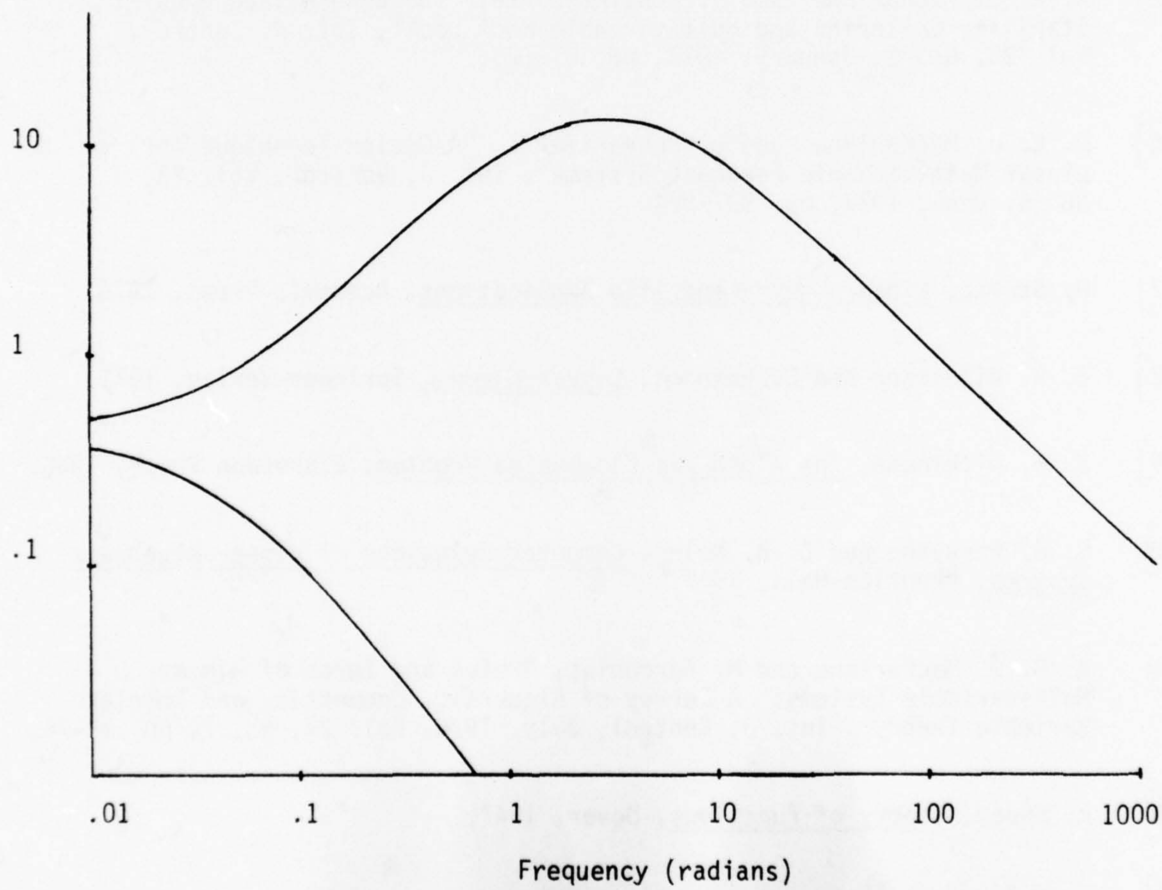

Fig. 10 Closed-Loop Frequency Response - $1/\sigma(I + G^{-1})$

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APPENDIX F

MULTIVARIABLE FEEDBACK DESIGN USING THE INVERSE NYQUIST ARRAY

APPENDIX F

MULTIVARIABLE FEEDBACK DESIGN USING THE INVERSE NYQUIST ARRAY

by J. C. Doyle

An analysis of a design presented by Rosenbrock[1] provides the framework for discussing the potential applicability of the diagonal dominance/Inverse Nyquist Array methods. Rosenbrock uses his techniques on an example problem which is presented for the sole purpose of demonstrating the obvious superiority of his methods. One might expect that under these circumstances, the diagonal dominance/Inverse Nyquist Array methods would look good. The following analysis shows that this is emphatically not the case.

The example Rosenbrock treats has the matrix transfer function

$$G(s) = \begin{bmatrix} \dfrac{1 - s}{(1 + s)^2} & \dfrac{2 - s}{(1 + s)^2} \\[2ex] \dfrac{1 - 3s}{3(1 + s)^2} & \dfrac{1 - s}{(1 + s)^2} \end{bmatrix}$$

---

1 "Design of Multivariable Control Systems Using the Inverse Nyquist Array", PROC. IEE, Vol. 110, No. 11, November 1969.

Unfortunately, since no physical system is given, we are left guessing as to what the model might mean. Note that all the dynamics are around 1 radian so we might conjecture that the validity of the model is limited to fairly low frequencies.

Rosenbrock applies his Inverse Nyquist Array/ diagonal dominance methods and comes up with the feedback gain matrix

$$K = 50 \begin{bmatrix} -2 & -6 \\ \frac{2}{3} & 3 \end{bmatrix}$$

Rosenbrock comments that the

> "...example is not a trivial one. The original system allows four different single control loops to be set up. All of these have nonminimum phase, and they all present considerably more difficulty than either of the two dominant loops. Attempts to set up two loops simultaneously around the original system are equally unpromising. Yet only a matrix of constant interconnections is needed, to allow two simultaneous loops of relatively good, and easily improved, performance. The method described allows this matrix to be obtained by a systematic procedure, which can take account of engineering constraints."

Let's analyze the performance of Rosenbrock's design. The closed loop system is

$$H(s) = \frac{1}{\rho_c(s)} \begin{bmatrix} 25\,(s^2 + 3s + 102) & 150s\,(s+1) \\ \frac{25\,(s+1)\,(3s+5)}{3} & 50\,(3s^2+4s+51) \end{bmatrix}$$

$$\rho_c(s) = s^3 + 178\,s^2 + 278\,s + 2601$$

with poles at

$$s = -.746 \pm 3.77\,j, \quad -176.5.$$

The pole locations should be an immediate cause for concern. There is a fairly oscillatory pair and one at -176.5 ??? Hmmm. Seems a might far out for a system that started out with poles at -1.

How about singular values? The σ-plot for $I + (KG)^{-1}$ is shown in Figure 1. (This is the margin plot for loops broken at the inputs. The minimum value of σ is a measure of the size of the perturbation which may be tolerated without instability.)

Note that the minimum σ is .1. Not so hot. Equally interesting is that the minimum σ stays below .5 all the way out to nearly 200 radians.

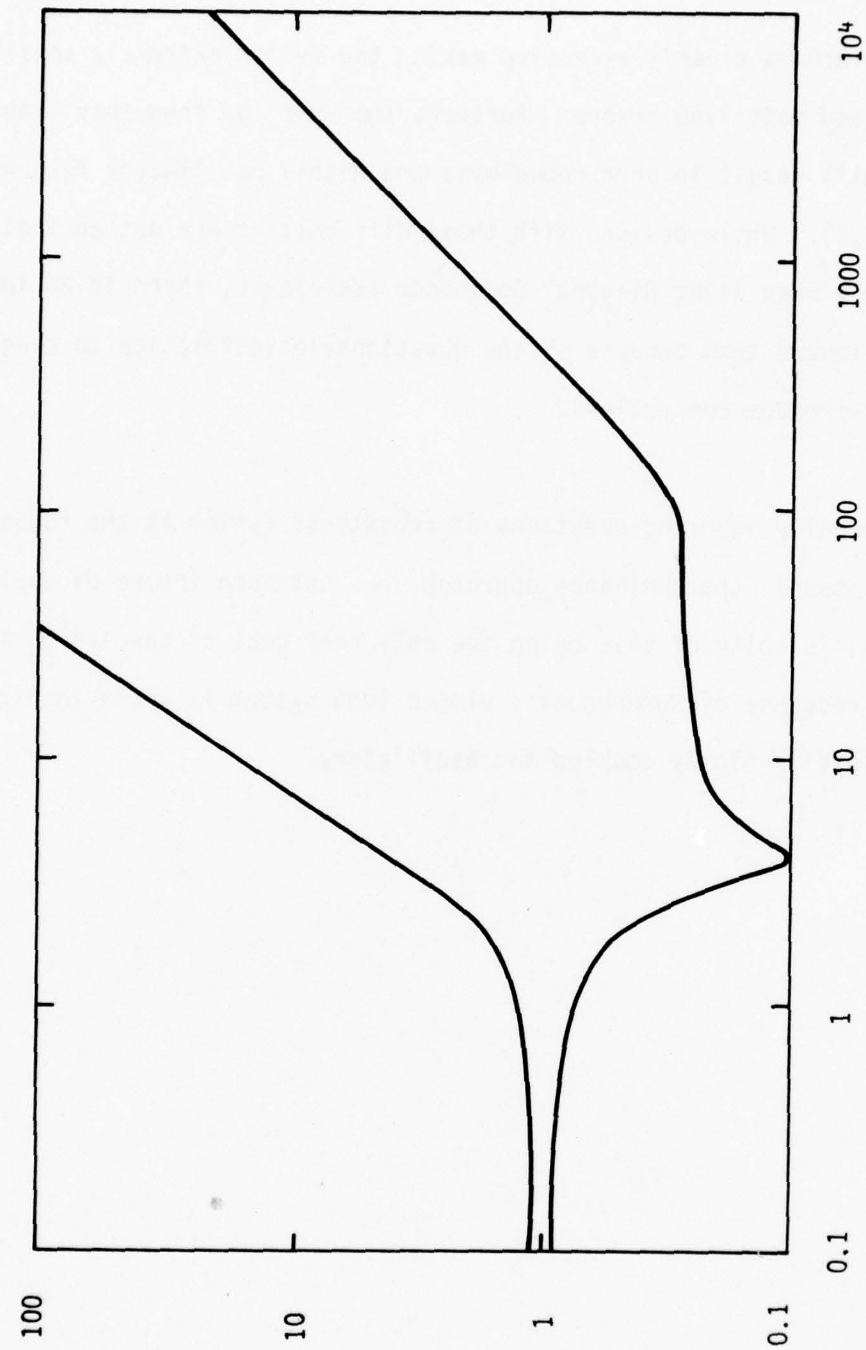


Figure 1. σ-Plot for INA Method

The bandwidth is clearly excessive making the system extremely sensitive to noise and modelling errors. Further, the poor low frequency stability margins will result in poor robustness and highly oscillatory responses. (See Fig. 2). While designs with these difficulties are not entirely unavoidable when using Diagonal Dominance techniques, there is an inherent tendency toward them because of the questionable restriction to single-degree-of-freedom controllers.

While generally ignoring questions of robustness (which is the fundamental feedback issue) the dominance approach does not even insure decoupled responses, in spite of this being the only real goal of the approach. The step response of Rosenbrock's closed loop system is shown in Fig. 2. The response is highly coupled and oscillatory.

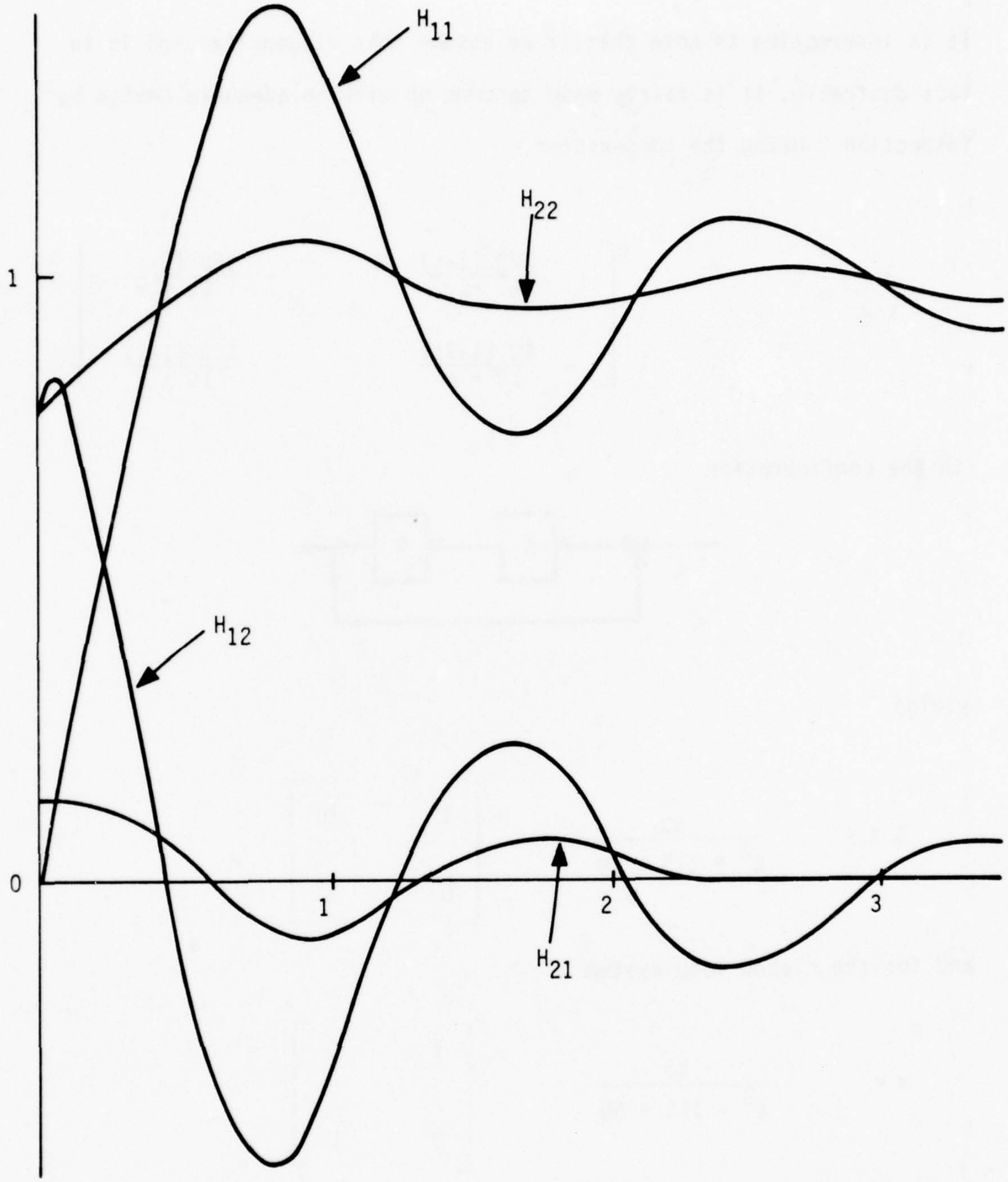
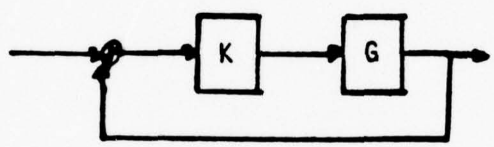


Figure 2. Step Response for INA Design

It is interesting to note that if we assume that diagonalization is in fact desirable, it is fairly easy to come up with an adequate design by inspection. Using the compensator

$$K = \begin{bmatrix} \frac{120\,(1-s)}{10+s} & -\frac{120\,(2-s)}{10+s} \\[2ex] -\frac{40\,(1-3s)}{10+s} & \frac{120\,(1-s)}{10+s} \end{bmatrix}$$

in the configuration



yields

$$G\,K = \frac{40}{s^2 + 11S + 10} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and for the closed loop system

$$H = \frac{40}{s^2 + 11S + 50} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Adding some feed forward to give unity steady-state response to a step, we get the step response in Figure 3. The σ-plot for $I + (KG)^{-1}$ is shown in Figure 4. It doesn't matter for my design if the loops are broken at inputs or outputs. Rosenbrock's σ-plot is shown by a dotted line in Figure 4 for comparison.

Beyond noting the obvious huge improvement in transient response and robustness, it's rather meaningless to attempt further comparison of the two designs without knowledge of exactly what physical system the model is intended to represent.

In summary, it seems clear that the diagonal dominance/INA approach does not live up to the claims made for it. The attempt to extend the classical frequency domain techniques to multivariable systems is laudable, but unfortunately the INA represents a very preliminary and primitive step in this extension. The classical frequency-domain techniques are attractive in that they make certain key characteristics of a SISO system readily apparent to the designer in a simple graphical form. The Inverse Nyquist Array seems to confuse and obscure the critical multivariable issues rather than clarify them.

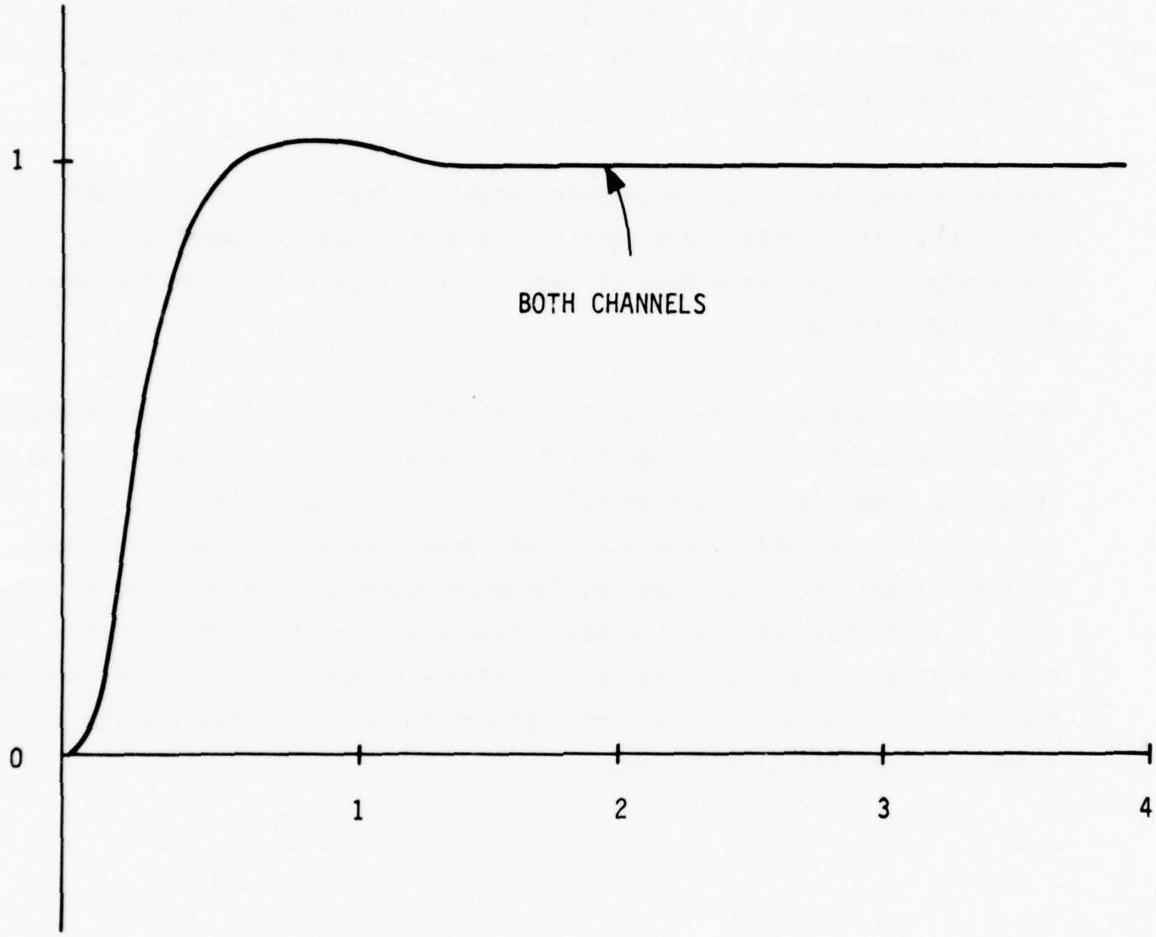


Figure 3. Step Response

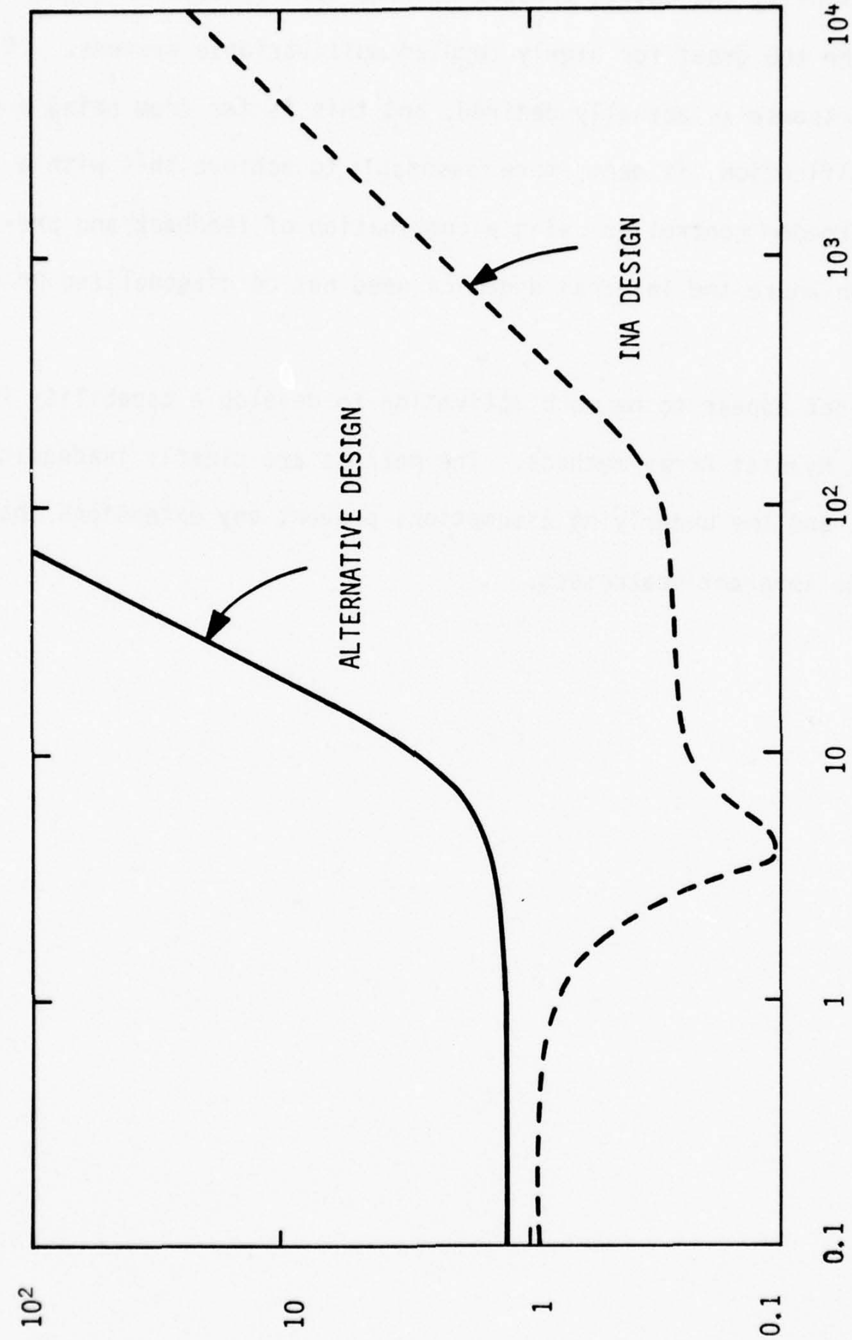


Figure 4. σ-Plot

There seems to be no point to diagonalization of the return difference matrix except to simplify the design procedure. The cost of such a diagonalization will often be too great for highly coupled multivariable systems. If a decoupled response is actually desired, and this is far from being a standard design specification, it seems more reasonable to achieve this with a two-degree-of-freedom controller using a combination of feedback and pre-compensation where the internal dynamics need not be diagonalized or dominant.

There does not appear to be much motivation to develop a capability in the Inverse Nyquist Array methods. The methods are clearly inadequate as they stand, and the underlying assumptions prevent any extensions that would overcome the inherent weaknesses.

APPENDIX G
MULTIVARIABLE FEEDBACK DESIGN USING CHARACTERISTIC LOCI

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MULTIVARIABLE FEEDBACK DESIGN USING CHARACTERISTIC LOCI

By J.C. Doyle

An interesting approach to multivariable feedback design is the characteristic loci (CL) methodology developed by A. G. MacFarlane. Reference [1] gives a thorough description of this approach so I won't repeat what's there. My conclusion from working some simple examples and from studying the properties of the CL approach is that it is inadequate for our needs and there seems to be little motivation to develop a capability in it.

We already know that the eigenvalues of the return difference matrix (which are used in CL) are not reliable measures of robustness (see example 2 in [2]). An equally serious difficulty with the CL approach is the tendency to very high bandwidth feedback loops. While this, of course, gives very fast responses on paper, it leads to unstable plants when all the dynamics and uncertainty are thrown in. This stems partially from the obsession of making the loop dynamics noninteracting. Even if a closed-loop decoupled response is desired, there is absolutely no need to diagonalize the loop dynamics unless there is some aversion to using precompensation. Feedback loop diagonalization constrains the solution enough that it is often impossible to achieve a robust feedback design.

This can be seen by considering an example design taken from [1] . The following is excerpted from [1] :

"The example used here to illustrate the use of the design technique is based on a model of an open-loop unstable chemical reactor. Its state-space description is given by the set of matrices

$$\left.\begin{aligned} A &= \begin{bmatrix} 1\cdot38 & -0\cdot2077 & 6\cdot715 & -5\cdot676 \\ -0\cdot5814 & -4\cdot29 & 0 & 0\cdot675 \\ 1\cdot067 & 4\cdot273 & -6\cdot654 & 5\cdot893 \\ 0\cdot048 & 4\cdot273 & 1\cdot343 & -2\cdot104 \end{bmatrix} \\ B &= \begin{bmatrix} 0 & 0 \\ 5\cdot679 & 0 \\ 1\cdot136 & -3\cdot146 \\ 1\cdot136 & 0 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{aligned}\right\}$$

corresponding to a transfer-function matrix

$$G(s) = \frac{1}{\Delta_o(s)} \begin{bmatrix} \Gamma_{11}(s) & \Gamma_{12}(s) \\ \Gamma_{21}(s) & \Gamma_{22}(s) \end{bmatrix}$$

where

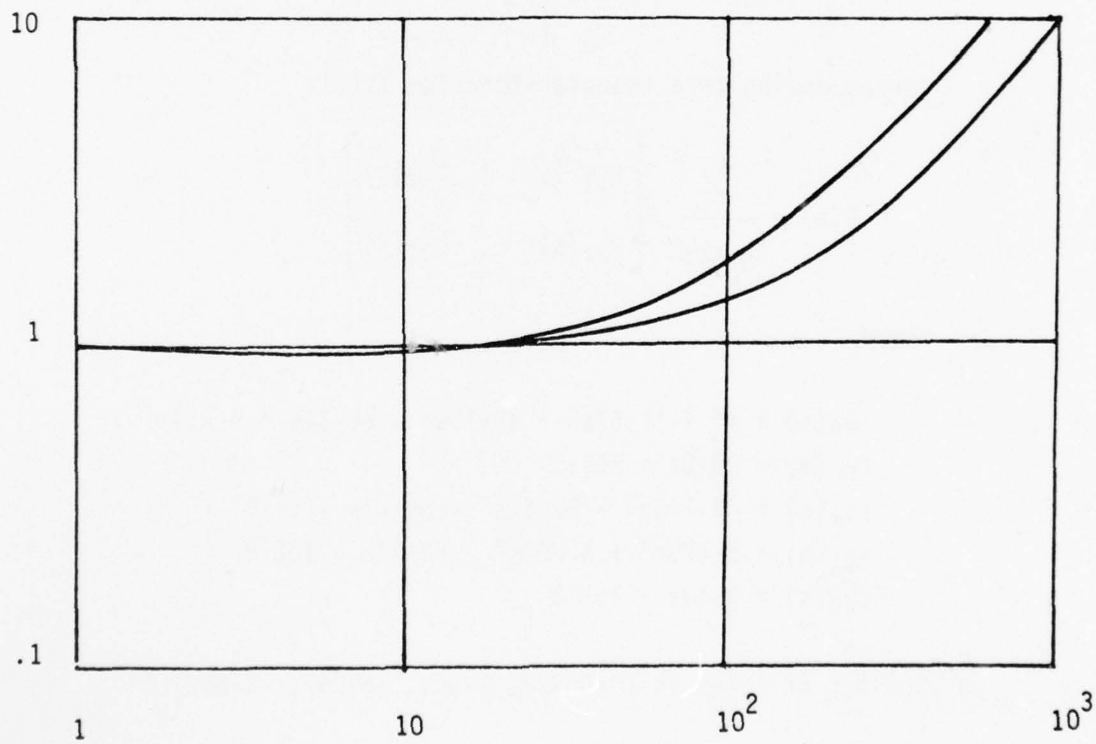
$$\Delta_o(s) = s^4 + 11\cdot67s^3 + 15\cdot75s^2 - 88\cdot31s + 5\cdot514$$
$$\Gamma_{11}(s) = 29\cdot2s + 263\cdot3$$
$$\Gamma_{12}(s) = -3\cdot146s^3 - 32\cdot62s^2 - 89\cdot83s - 31\cdot81$$
$$\Gamma_{21}(s) = 5\cdot679s^3 + 42\cdot67s^2 - 68\cdot84s - 106\cdot8$$
$$\Gamma_{22}(s) = 9\cdot43s + 15\cdot15$$

The set of poles is $\{0\cdot06318, 1\cdot991, -5\cdot057, -8\cdot666\}$."

The design which is finally arrived at using the characteristic loci approach has a feedback gain matrix of

$$K(s) = \begin{bmatrix} 0 & \dfrac{20s + 18}{s} \\ -\dfrac{(20s + 42)}{s} & -\dfrac{16}{s} \end{bmatrix}$$

The closed-loop poles are at $\{-1\pm.07j, -2.68, -4.03, -62.5, -117\}$. The poles at -62.5 and -117 suggest excessive bandwidth and the singular values of $I + (KG)^{-1}$ confirm this as shown below.



Note that both singular values are below 2 at 100 radians. It's very hard to believe that a chemical reactor would be modelled that accurately at such high frequencies when the four poles in the model are all smaller than 10 radians. A couple of extra unmodelled poles at around -50 and this system is closed-loop unstable.

SUMMARY

The characteristic loci approach represents a sophisticated attempt at extending frequency domain techniques to the multivariable setting. Some interesting ideas have come out of this approach, including the notion of an approximately commutative compensator and the use of eigenvalues of the return ratio matrix to characterize the system. Unfortunately, eigenvalues are not the right quantities to study as has been amply demonstrated. The major drawback to the practical use of the CL approach beyond the inadequacy of eigenvalues is the requirement of extremely high feedback loop bandwidths. This requires that there be virtually no uncertainty, in which case feedback is unnecessary.

The rather complete neglect of robustness issues make the CL approach inadequate for feedback design and the limitations are so intimately tied with the methodology that useful extensions appear unlikely. However, the literature surrounding the characteristic loci methods is rich in mathematical techniques and insights.

A further comment which applies to both the INA and the CL methods is that they are limited to square systems. Our results in advanced filter/observer design techniques show that additional measurements are very useful for

improving robustness. It is clear that having to square down the system before the design begins is a limitation.

REFERENCES

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2 J. C. Doyle, "Robustness of Multiloop Linear Feedback Systems". (See Appendix E.)

APPENDIX H

SINGULAR VALUES AS FUNCTIONS OF A COMPLEX VARIABLE

APPENDIX H

SINGULAR VALUES AS FUNCTIONS OF A COMPLEX VARIABLE

By J. C. Doyle

INTRODUCTION

While the robustness results using singular values concentrates on matrix transfer functions $Q(s)$ where $s = j\ \omega$, additional insight may be gained by considering the singular values of $Q(s)$ where s ranges over the entire complex plane. (For our purposes $Q(s)$ may be taken as the loop transfer function $G(s)$, $I + G(s)$, $I + G(s)^{-1}$, or whatever transfer function is of interest). This is similar to the approach taken by MacFarlane and Postlethwaite [1], except that they consider the eigenvalues of $Q(s)$ rather than its singular values. The development which follows parallels closely that in [1], with the exception being that the eigenvalues of $Q(\bar{s})^T\, Q(s)$ rather than of $Q(s)$ are considered where $\bar{s}$ indicates the conjugate of s.

PRELIMINARIES

Let $Q(s)$ be an mxm rational transfer function. For any specific value of s, say s_0, the matrix $Q(s_0)$ will have a set of singular values $\{\sigma(s_o) | i=1, \ldots m\}$. Thus, the singular values of $Q(s)$ may be thought of as functions of the complex variable, s. The properties of these functions form the topics of this discussion.

The characteristic equation which defines the singular values is

$$\det\left[\alpha I - Q(\bar{s})^T Q(s)\right] \triangleq \nabla(\alpha,s,\bar{s}) = 0 \tag{1}$$

Where α is the square of the singular values. We will concentrate on α and remember that $\alpha = \sigma^2$.

Assuming that $\nabla(\alpha,s,\bar{s})$ is irreducible over the field of rational functions of s and $\bar{s}$, we may write

$$\nabla(\alpha,s,\bar{s}) = \alpha^m + p_1(s,\bar{s})\alpha^{m-1} + \ldots p_m(s,\bar{s}). \tag{2}$$

where $p_i(s,\bar{s}) = p_i(\bar{s},s)$ for all $i = 1,\ldots m$. (If $\nabla(\alpha,s,\bar{s})$ is reducible, the argument applies to each of its factors.)

If $b_0(\bar{s},s)$ is the least common denominator of the coefficients $\{p_i(s,\bar{s}): i=1,\ldots m\}$, (2) may be put in the form

$$b_0(s,\bar{s})\,\alpha^m + b_1(s,\bar{s})\,\alpha^{m-1} + \ldots b_m(s,\bar{s}) = 0 \tag{3}$$

where $b_i(s,\bar{s}) = b_i(\bar{s},s)$ for all $i=1,\ldots m$. Following [1], it may be shown that

$$b_0(s,\bar{s}) = |\Psi(s)|^2 \tag{4}$$

and

$$b_m(s,\bar{s}) = |\phi(s)|^2 \tag{5}$$

where $\Psi(s)$ and $\phi(s)$ are the pole and zero polynomials, respectively, of $Q(s)$.

It is now obvious that α is not an algebraic function of s due to the presence of the $\bar{s}$ in the coefficients of (3). Thus it is difficult to

make precise the exact nature of the multivalued map from $s \varepsilon C$ to $\alpha \varepsilon R$ defined by (3). In particular, the use of analytic continuation is inadequate to describe a map from a domain forming a Riemann surface, since the corresponding branches of the multivalued map in (3) are not locally analytic.

Because of this, the following results are not mathematically rigorous and additional work is needed to make them so. Nevertheless, the qualitative insights available are useful and worth considering.

Properties of Singular Values of $Q(s)$

Let $\alpha(s)$ be the multivalued function defined by the roots of (3). Then $\alpha(s)$ will have in general m distinct roots. An exception occurs only if (a) $b_0(s,\bar{s}) = |\Psi(s)|^2 = 0$, because the degree of (3) is lowered, or if (b) equation (3) has multiple roots.

DEFINITIONS

1. Ordinary Points of $\alpha(s)$

   An ordinary point of $\alpha(s)$ is any point in the complex plane such that neither (a) nor (b) is true.

2. Critical Points of $\alpha(s)$

   A critical point of $\alpha(s)$ is any point in the complex plane such that either (a) or (b) is true or both (a) and (b) are true.

3. Branch Point of $\alpha(s)$

A branch point of $\alpha(s)$ is any point in the complex plane such that (b) is true.

Every point in the complex plane is either an ordinary point or a critical point. The function $\alpha(s)$ may be thought of as a set of m functions each defined on the complex plane punctured by the critical points. Each of these (non-unique) functions will be called a branch of $\alpha(s)$. The arguments used by Knopp 2 to prove that the branches of algebraic functions are continuous may be extended to prove that the branches of $\alpha(s)$ are continuous. However, as mentioned before, the branches are not analytic functions so the technique of analytic continuation may not be used to redefine the domain of the functions. Thus, while we may talk of a Riemann surface as constituting the domain of $\alpha(s)$, this notion has not been made precise.

In view of (3), (4), and (5) we may state the following:

1. If s_c is a closed-loop pole of the system $\left[I + G(s)^{-1}\right]^{-1}$, then

$$\lim_{s \to s_c} \underline{\sigma}\left[I + G(s)\right] = 0$$

and

$$\lim_{s \to s_c} \underline{\sigma}\left[I + G^{-1}(s)\right] = 0$$

2. If s_o is an open-loop pole of the system $G(s)$, then

$$\lim_{s \to s_o} \underline{\sigma}\left[I + G(s)\right] = \infty$$

3. If z is a transmission zero of G(s), then

$$\lim_{s \to z} \bar{\sigma}\left[I + G^{-1}(s)\right] = \infty$$

4. More generally, if s_0 and z are poles and zeroes, respectively, of Q(s), then

$$\lim_{s \to z} \underline{\sigma}\,(Q(s)) = 0$$

and

$$\lim_{s \to s_0} \bar{\sigma}\,(Q(s)) = \infty.$$

A number of qualitative observations concerning the map $\sigma(s) = (\alpha(s))^{1/2}$ may be made.

1. Poles and zeroes of Q(s) which are near each other on the complex plane will not tend to cancel unless they appear on the same branch of $\alpha(s)$.
2. If a closed loop pole of $\left[I + G(s)^{-1}\right]^{-1}$ is close to the imaginary axis, there will be a tendency for $\underline{\sigma}\,(I + G(jw))$ and $\underline{\sigma}\,(I + G(jw)^{-1})$ to be small near that pole unless it is canceled by a pole of I+G(s) or $I+G^{-1}(s)$.
3. As shown in [3], having all the zeroes of Q(s) far from the imaginary axis does not insure that $\underline{\sigma}(Q(j\omega))$ may not be small. While

it is true that

$$\prod_{i=1}^{m} \sigma_i(s) = \left| \frac{\phi(s)}{\Psi(s)} \right|$$

where ϕ and Ψ are the zero and pole polynomial respectively of $Q(s)$, it may that $\underline{\sigma}(j\omega) << \overline{\sigma}(j\omega)$ for some ω and $\underline{\sigma}$ may be small even though $\prod_{i=1}^{m} \sigma_i$ is large.

4. Consider individually each branch of $\sigma(s) = (\sigma(s))^{\frac{1}{2}}$ as a function on the punctured complex plane. We may imagine each branch as a contour where the height of the contour at a point s is the value of the branch function at that point. The robustness properties of a system then depend on the heights of the contours of the branches of $\sigma(I\_G(s))$ or $\sigma(I\_G(s)^{-1})$ along the imaginary axis.

Additional research is needed to make precise the notions presented here. We anticipate that the use of singular values over the entire complex plane will prove to be a useful extension to the notion of a scalar transfer function defined on the complex plane.

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2 K. Knopp, Theory of Functions, Part 2, New York: Dover, 1947.

3 J. C. Doyle, "Robustness of Multiloop Linear Feedback Systems." (See Appendix E.)

APPENDIX I

ROBUSTNESS WITH OBSERVERS

APPENDIX I

ROBUSTNESS WITH OBSERVERS

by J. C. Doyle and G. Stein

ABSTRACT

This paper describes an adjustment procedure for observer-based linear control systems which asymptotically achieves the same loop transfer functions (and hence the same relative stability, robustness, and disturbance rejection properties) as full-state feedback control implementations.

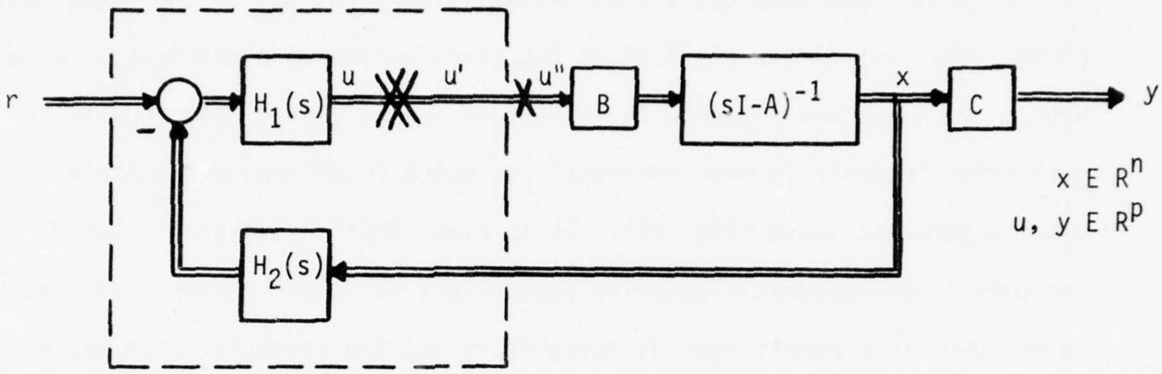
I. Introduction

The trouble with observers is that they tempt us, through the expedient of state reconstruction, to assign undue generality to control results proven only for the full-state feedback case. An example is the recent robustness result of Safonov and Athans [1]. This result shows that multivariable linear-quadratic optimal regulators have impressive robustness properties, including guaranteed classical gain margins of -6 db to + ∞ db and phase margins of $\pm$ 60 deg. in all channels. The result is only valid, however, for the full state case. If observers or Kalman filters are used in the implementation, no guaranteed robustness properties hold. In fact, a simple example has shown that legitimate LQG controller-filter combinations exist with arbitrarily small gain margins in both the positive and negative db direction [2].

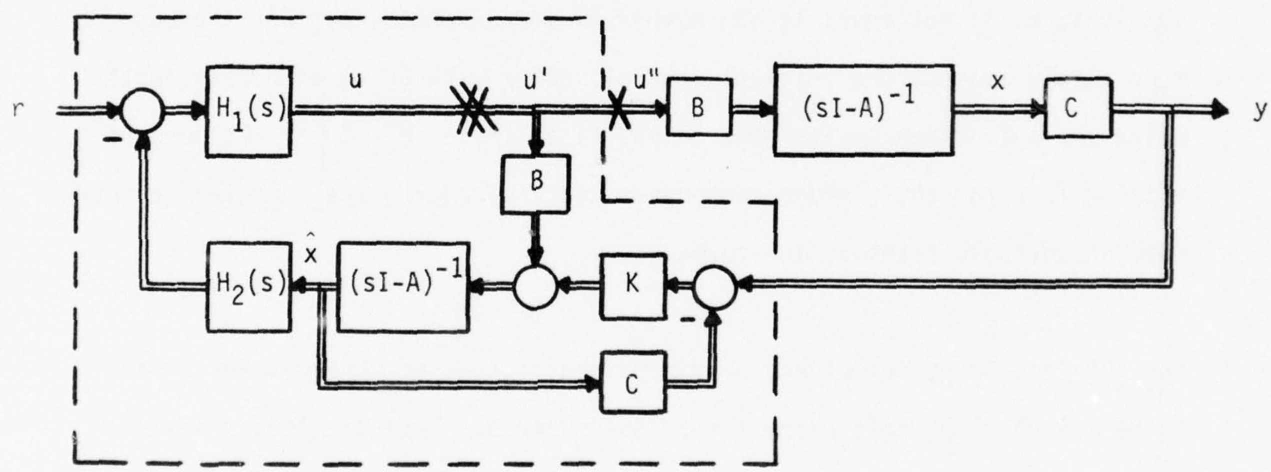
In light of these observations, the robustness properties of control systems with filters or observers need to be separately evaluated for each design. Moreover, because such evaluations can come up with embarassingly small margins, a "design adjustment procedure" to improve robustness would be very desirable. The present paper provides such a procedure. We show that while the commonly suggested approach of "speeding-up" observer dynamics will not work in general, alternate procedures which drive some observer poles toward stable plant zeros and the rest toward infinity do achieve the desired objective. In effect, full-state robustness properties can be recovered asymptotically if the plant is minimum phase. This occurs at the expense of noise performance.

The principal results of the paper are summarized in Section II, where we introduce and interpret certain transfer function properties of observer-based control systems, and in Section III, where we develop the "adjustment procedure". A simple example which illustrates these results is given in Section IV.

Figure 1. Linear Multivariable Control Loop



1A. Full-State Feedback Implementation



1B. Observer-Based Implementation

II. Transfer Function Properties of Observer-Based Controllers

We consider the general multivariable control loop illustrated in Figure 1. The plant is an n-th order linear system, both observable and controllable, with m inputs, p=m outputs, and no transmission zeros [3] in the right half plane. The control law consists of two transfer function matrices $H_1(s)$ and $H_2(s)$. H_2 is driven either with full-state feedback (Fig. 1A) or with an n-th order "model-reference observer" [4] which reconstructs the state in the usual asymptotic sense (Fig. 1B). It is clear that this overall control loop includes linear-quadratic-gaussian controllers as special cases. It also allows dynamic elements such as integrators and lag elements which may be required in more realistic control situations.

This configuration also applies to nonsquare plants for which the number of controls, m, is not equal to the number of measurements, p. For the case, m<p, simply augment the original control vector with (p-m) more components which are not driven by the controllers (i.e., $\left[H_1^T = H_{11}^T \vdots 0\right]$). Columns of the B matrix for these added components must, of course, be selected to introduce no unstable transmission zeros.

For the case, m>p, select any p- dimensional subset of controls for which there are no right half plane transmission zeros. Then the loop transfer properties which are established in this paper apply to this p-dimensional subset of control loops, with the remaining (m-p) loops closed.

A dashed line is shown in both Figure 1A and 1B in order to distinguish between elements of the loop which are part of the controller and those

which are part of the plant. Since we design and implement the controller, there is relatively little uncertainty associated with it, whereas there may be significant differences between the actual plant and its model. The loop transfer functions which we examine for robustness,below, are then taken with respect to the loop breaking point, X, at the control signal interface between these two sets of elements. Very misleading robustness results can be obtained for alternate loop breaking points, for example point XX. This is also shown below.

The following properties can be established for the above two control loop implementations:

Property 1

The closed loop transfer function matrices from command r to state x are identical in both implementations.

Property 2

The loop transfer function matrices from control signal u' to control signal u (loops broken at point XX) are identical in both implementations

Property 3

The loop transfer function matrices from control signal u" to control signal u' (loops broken at point X) are generally different in the two implementations even when the observer's error dynamics are allowed to be arbitrarily fast.

Property 4

The loop transfer function matrices from u" to u' are identical in both implementations if the observer dynamics satisfy

$$K \left[I + C\,(sI-A)^{-1}K\right]^{-1} = B\left[C(sI-A)^{-1}B\right]^{-1} \qquad (1)$$

for all values of the complex variable s. The A, B, and C's above are plant matrices and K is the observer gain.

The first two of these properties are very well known [5,6]. They can be easily verified by noting that the transfer functions from u' to x and from u' to $\hat{x}$ are identical because the nominal error dynamics of the observer are not controllable from u'. These two properties are also the source of much of the temptation surrounding observers, however. We see that input/output properties are the same and even certain loop transfer functions are the same. The latter promise equal relative stability properties, equal tolerance to uncertainties (robustness), and equal disturbance rejection properties. What more could we ask for?

The problem, of course, is that the loop transfer properties are the same at Point XX, inside our own control implementation where only masochists would insert significant uncertain elements or disturbances. According to Property 3, equal loop transfer characteristics are not obtained at the control signal interface to the plant, Point X, where Nature gets to insert uncertainties and disturbances. It is at this point that robustness properties must be measured, and, as seen in [2], it is here that observer-

based implementations can fall well short of our objectives.

Property 3 will be verified by means of example later. We now turn instead to the more interesting Property 4. This result is apparently not as well known, so a simple derivation is given in Addendum 1. It is important because it offers a way to adjust observers so that full-state loop transfer characteristics are recovered at Point X. In particular, suppose the observer gains are parameterized as a function of a scalar variable q. Let this function, K(q), be selected such that as $q \to \infty$

$$K(q) \quad \to \quad q\ BW \tag{2}$$

for any nonsingular matrix W. Then equation (1) will be satisfied asymtotically as $q \to \infty$. The resulting observer error dynamics will have limiting poles given by roots of the polynominal

$$\chi(s) = \det(sI-A)\det\left[I + qC(sI-A)^{-1}BW\right]. \tag{3}$$

P of these roots will tend toward the P finite transmission zeros of the plant (stable by assumption) and the rest will tend to infinity. It is clear from this that the commonly suggested approach of making all roots of the error dynamics arbitrarily faster is generally the wrong thing to do.

III. An Observer-Adjustment Procedure

Equation (2) defines the required limiting characteristics of an adjustment trajectory, $K(q)$, which changes arbitrary initial nominal observer gains, $K(o)$, with poor robustness properties into better gains asymptotically. We still need to define details of such trajectories.

A basic requirement for every point of an adjustment trajectory is stability of the observer error dynamics. Clearly, if we violate this requirement, overall closed loop stability is also lost. (Note that this does not mean that the net compensator within the dashed lines of Figure 1B needs to be stable.) One way to assure stable error dynamics is to restrict the observer to be a Kalman filter for some set of noise parameters. That is, let

$$K(q) = \Sigma(q)\ C^T R^{-1} \tag{4}$$

with $\Sigma(q)$ defined by the Riccati equation

$$A\Sigma + \Sigma A^T + Q(q) - \Sigma C^T R^{-1} C\Sigma = 0 \tag{5}$$

As usual we take $Q = Q^T \geq 0$ and $R = R^T > 0$. For Kalman filters, these matrices represent given process noise and measurement noise intensities, respectively. Here they are treated more freely as design parameters which we can select to suit broader purposes. In particular, let

$$Q(q) = Q_o + q^2 BVB^T \tag{6}$$

$$R = R_o \tag{7}$$

where Q_o and R_o are noise intensities appropriate for the nominal plant, and V is any positive definite symmetric matrix. With these selections,

the observer gain for $q = o$ corresponds to the nominal Kalman filter gain. However, as q approaches infinity, the gains are seen from (5) to satisfy,

$$K\ R\ K^T \longrightarrow q^2\ BVB^T$$

and

$$K \longrightarrow q\ B\ V_1\ R_1^{-1}, \tag{8}$$

where V_1 denotes some square root of V ($V_1 V_1^T = V$) and, similarly, R_1 is some square root of R. Since (8) is a special case of (2), it follows that the adjustment procedure defined by (4)-(7) will achieve the desired robustness-improvement objective.

Note that the second term in equation (6) can be interpreted as extra process noise added directly to the control input of the plant. Within the constraints of Kalman filter mathematics, such "fictitious noise" is a natural mechanism to represent uncertainties at this point of the control loop. It is nice to know that the resulting filter design actually responds with a corresponding robustness improvement. Note, however, that arbitrary increases of the existing noise matrix (i.e., $Q = (1 + q^2)\ Q_o$) or addition of arbitrary full rank noise process (i.e., $Q = Q_o + q^2 W$ with $W = W^T > o$) which are often suggested as other intuitive robustness improvement methods, will not in general produce the desired effect.

Finally, we note that the use of Kalman filter equations in the adjustment procedure is not fundamental. The filters merely provide a convenient way to assure stability along the entire adjustment trajectory. Any other procedure (pole placement, for example) with the same limiting properties (2) could be used as well.

IV. An Example

To illustrate the observer properties and adjustment procedure above, consider the following example:

Plant:

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 35 \\ -61 \end{bmatrix} \xi \tag{9}$$

$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} x + \eta \tag{10}$$

with $E(\xi) = E(\eta) = 0; \quad E\left[\xi(t)\xi(\tau)\right] = E\left[\eta(t)\eta(\tau)\right] = \delta(t-\tau)$

Controller:

$$u = \begin{bmatrix} -50 & -10 \end{bmatrix} \hat{x} + \begin{bmatrix} 50 \end{bmatrix} r \tag{11}$$

The plant in this example is a (harmless) stable system with transfer function.

$$\frac{y}{u}(s) = \frac{s + 2}{(s + 1)(s + 3)} \tag{12}$$

The controller happens to be a linear-quadratic one, corresponding to the performance index

$$J = \int_0^\infty (x^T H^T H x + u^2)\, dt \tag{13}$$

with

$$H = 4\sqrt{5} \begin{bmatrix} \sqrt{35} & 1 \end{bmatrix}$$

It places the closed loop regulator poles at

$$s = -7.0 \pm j2.0$$

A Nyquist diagram (polar plot of the loop transfer function at Point X) for the full-state design is given in Figure 2. Gain margin is infinite in both directions and there is over 85^{0} phase margin. The design is then implemented using a Kalman filter for the given noise parameters. The Nyquist plot for the resulting observer-based controller is also shown in Figure 2. Oops... less than 15^{0} phase margin.

In an effort to improve this margin, one adjustment to the filter that could be made is to speed it up. So, we can try moving the filter/observer poles to the left in a second-order Butterworth pattern. For the filter/observer poles at $-22 \pm 17.86j$ one gets the third Nyquist plot in Figure 2. As can be seen, the results are less than satisfactory. Not only are the margins disappearing (now less than 10 degrees) but the loop bandwidth has increased (crossover has gone from approximately 12. to 40. rad/sec).

Unless we're trying to design an explosive device, this is clearly undesirable. It gets worse as the filter gets faster. In fact, it can be shown that the margins go asymptotically to zero for large gains, while the loop bandwidth goes to infinity. The present example is not a pathological one, either. Similarly undesirable characteristics for fast filters are obtained with most systems.

When the observer adjustment procedure of Section III is applied to the same example, much more pleasing behavior is obtained. Following (6)-(7), we let the process noise covariance matrix be

$$Q = \begin{pmatrix} 35 \\ -61 \end{pmatrix} (35 \quad -61) + q^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0 \quad 1) \quad . \tag{14}$$

We then increase q from zero until a reasonable compromise between noise performance and robustness is achieved. Some results of this process are summarized in Figure 3 and Table 1. Figure 3 shows Nyquist diagrams for q^2 = 100, 500, 1000, and 10,000. Margins improved with essentially no change in bandwidth as the modified loop transfer function tends toward full state optimal. Noise performance is summarized in Table 1 for the same set of q values. As expected, the error covariance of the adjusted filter with respect to the original noise increases markedly with q. However, there was not the same deterioration in state covariance.

Table 1 also documents other parameters associated with these design points - - poles of the error dynamics, margins, and filter gains. Note in particular that the filter poles tend toward the plant zero and toward infinity, as required by (3).

This adjustment procedure was also successfully applied to reconstruction of measured outputs after sensor failures for the A-7D aircraft. [8] In this application the optimum Kalman filter produced an unstable system when tested in hybrid simulation over the A-7D flight envelope. After attempts with "ad hoc" fictitious noise adjustment procedures failed the method discussed here successfully stabilized the system. Also, the resulting error covariance properties remained closed to the optimum values.

V. Conclusions

This paper illustrates some of the difficulties one can get into by relying on observers for state reconstruction. We have concentrated on robustness properties. In general, these will be poorer for observer-based implementations than for full-state implementations. For minimum-phase systems, however, full-state robustness can be recovered asymptotically provided it is done correctly. Fast observers are not in general correct. A "fictitious noise" adjustment procedure was suggested which is.

The apparent practical value of this procedure is that it gives a simple way of trading off between noise rejection and margin recovery. When $q = 0$, the filter will be optimal with respect to the "true" (as modelled) system noise. As q increases the filter will do a poorer job of noise rejection but the closed-loop stability margins will improve. Hopefully, a satisfactory compromise can be found through the adjustment of the single parameter q. We stress that margin recovery occurs at Point X in Figure 1 -- at the control signal interface to the outside world. Asymptotically, the full-state and observer-based implementations will have the same tolerance to disturbances and uncertain elements inserted at this point. While Point X is clearly a physically important one (more important than Point XX, certainly), engineers who may wish to test robustness at still other points in the control loop should recognize that the recovery results may not be applicable there.

The suggested adjustment procedure is essentially the dual of a sensitivity recovery method suggested by Kwakenaak [7]. The latter provides a method

for selecting the weights in the quadratic performance index so that full-state sensitivity properties are achieved asymptotically as the control weight goes to zero. In this case, however, closed loop plant poles instead of observer poles are driven to the system zeros, which can result in unacceptable closed loop transfer function matrices for the final system.

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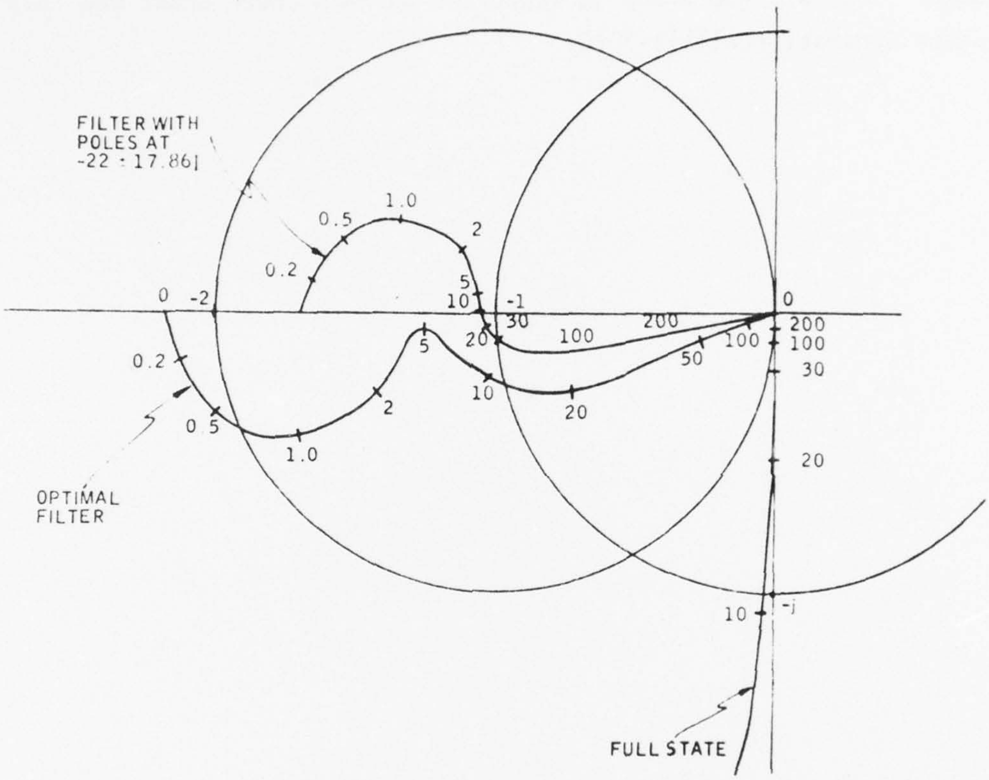

Figure 2. Loop Transfer Functions of Example:
"Fast Filter" Adjustment Procedure

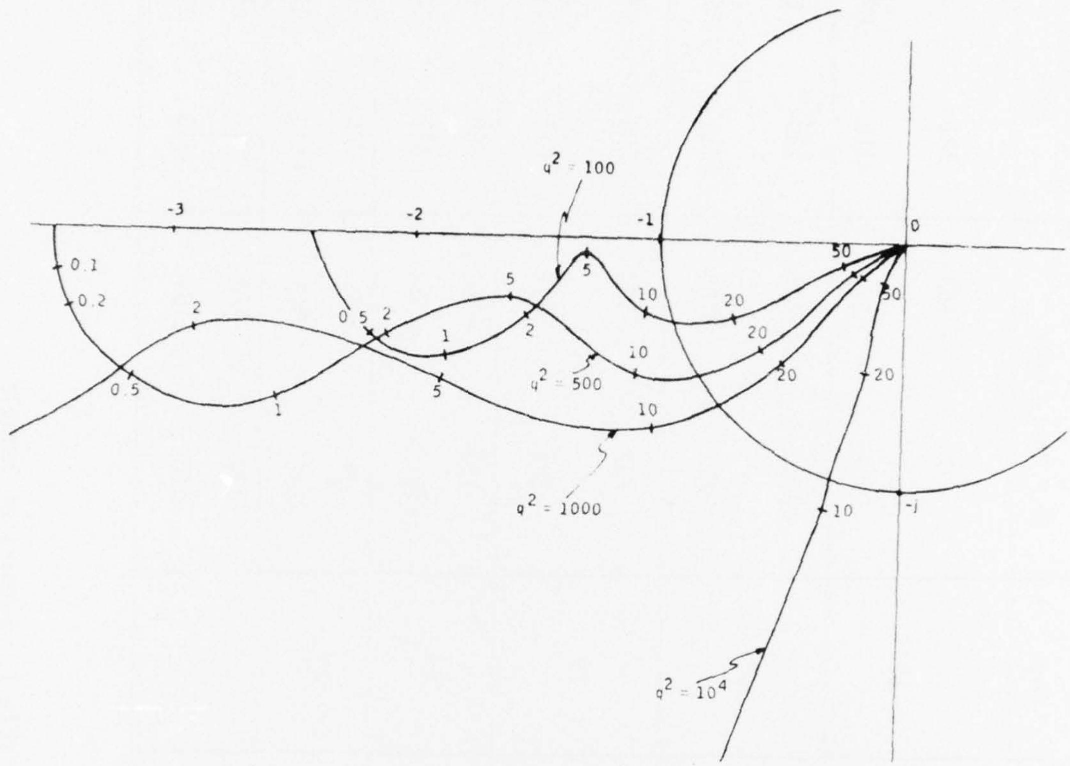

Figure 3. Loop Transfer Function of Example: "Fictitious Noise" Adjustment Procedure

	FILTER POLES	GAIN MARGIN db	PHASE MARGIN deg	ERROR COVARIANCE $E\left[(x-\hat{x})(x-\hat{x})^T\right]$		STATE COVARIANCE $E(xx^T)$		FILTER GAIN
Optimal LQG Design	$-7 \pm 2j$	- 6.75	15	97	- 163	221	- 613	30
				- 163	277	-613	2070	- 50
Fast Filter Adjustment Procedure	$-22 \pm 17.86j$	- .98	<10	6284	-12224	130	- 613	720
				-12224	23788	-613	8517	-1400
Fictitious Adjustment Procedure $q^2 = 100$	-4.3	- 7.73	19	107	- 184	236	-613	26.8
	-13.1			- 184	319	-613	1812	- 40.2
$q^2 = 500$	-2.9	-10.9	33	163	- 301	268	-613	20.4
	-24			-301	564	-613	1497	- 17.7
$q^2 = 10^3$	-2.5	-13.9	42	204	-385	285	-613	16.7
	-33			-385	743	-613	1360	- 1.9
$q^2 = 10^4$	-2.1	-37	74	290	-570	317	-613	6.9
	-100			-570	1169	-613	1198	84.6

Table 1. Summary of Example

ADDENDUM 1: DERIVATION OF PROPERTY 4

Referring to Figure 1A, the loop transfer function from u" to u' of the full state implementation is obtained from the relationships

$$x = \Phi(Bu'' + Fv) \tag{A.1}$$

$$u' = -H_1H_2x \ , \tag{A.2}$$

where

$$\Phi = (sI-A)^1 \tag{A.3}$$

$$v = -G_1G_2x \ . \tag{A.4}$$

The variables v above are now shown in Figure 1 for the sake of simplicity. They denote the (m-p) control components for which loops are not broken in the event that $p < m$. Matrices F, G_1, and G_2 are to control input matrix and the feedback compensator matrices for these components, respectively. If the original plant is square or can be made square by augmenting (p-m) additional control variables, then v, F G_1 and G_2 are zero identically. For either situation, (A.1) - (A.4) define the following full-state loop transfew function:

$$u' = -H_1H_2 \ (I + \Phi FG_1G_2)^{-1} \ \Phi Bu'' \tag{A.5}$$

The corresponding relationships for observer-based implementations are (Fig. 1B).

$$\hat{x} = (\phi^{-1} + KC)^{-1} \ \{Bu' + Fv + KC\Phi(Bu'' + Fv)\}$$

$$= (\Phi^{-1} + KC)^{-1} \{Bu' + KC\Phi Bu'' + (\Phi^{-1} + KC)\ \Phi Fv\}$$

$$= (\Phi^{-1} + KC)^{-1} \{Bu' + KC\Phi Bu''\} + \Phi Fv \tag{A.6}$$

with

$$u' = -H_1H_2\hat{x}$$

$$v\ = -G_1G_2\hat{x} \tag{A.7}$$

This gives

$$u' = -H_1H_2(I + \Phi FG_1G_2)^{-1}\ (\Phi^{-1} + KC)^{-1}\ \{Bu' + KC\Phi Bu''\}. \tag{A.8}$$

Now applying the Matrix inversion lemma to the $(\Phi^{-1} + KC)^{-1}$ term in this expression gives

$$u' = -H_1H_2(I + \Phi FG_1G_2)^{-1} \left[\Phi - \Phi K(I + C\Phi K)^{-1}C\Phi\right] \{Bu' + KC\Phi Bu''\}$$

$$= -H_1H_2(I + \Phi FG_1G_2)^{-1}\ \ \Phi\left[B - K(I + C\Phi K)^{-1}C\Phi B\right]u'$$

$$-H_1H_2(I + \Phi FG_1G_2)^{-1}\ \ \Phi K\ (I + C\Phi K)^{-1}\ C\Phi B\ u''. \tag{A.9}$$

From (A.9) it follows that if (1) is satisfied, then the u' term on the right hand side vanishes and the u" term is identical to (A.5). Since u" is arbitrary, this establishes the claimed equality of loop transfer functions.

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